MATH 281C SPRING 2020 HOMEWORK 4

Target date: Tuesday(?), June ?, 2020

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do not re-prove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1 page) Work in ZFC. HW 3 Problem 6 shows how to force a Kurepa tree in a very easy way, at the price that we make use of an inaccessible cardinal. However, an inaccessible cardinal is not needed for a forcing construction of a Kurepa tree.

Define a poset $\mathbb{P}$ as follows.

- **Conditions** are ordered pairs $p = (u_p, f_p)$ where
  - $u_p = (\alpha_p, <_p)$ is a tree for some $\alpha_p < \omega_1$ such that for any $\beta < \beta' < \mathsf{ht}(u_p)$ and any $t \in \mathsf{Lev}_\beta(u_p)$ there is some $t' \in \mathsf{Lev}_{\beta'}(u_p)$ with $t <_p t'$.
  - $f_p : a \to B_p$ is an injective function where B_p be the set of all cofinal branches through u_p and $a \subseteq \omega_2$ is countable.
- **Ordering** is defined as follows: $p \le q$ iff
  - u_p end-extends u_q. More technically, $\alpha_q = \{t \in \alpha_p \mid \mathsf{ht}_{u_p}(t) < \mathsf{ht}(u_q)\}$ and $<_q = <_p \cap (\alpha_q \times \alpha_q)$.
  - $\mathrm{dom}(f_p) \supseteq \mathrm{dom}(f_q)$.
  - $f_q(\xi) = f_p(\xi) \cap \alpha_q$ whenever $\xi \in \mathrm{dom}(u_q)$.

Prove the following.

(a) $\mathbb{P}$ is ω_1-closed.
(b) Assuming CH, the poset $\mathbb{P}$ is ω_2-c.c.

Let G be a $(\mathbb{P}, \mathbf{V})$-generic filter. Then

(c) $\bigcup\{\alpha_p \mid p \in G\} = \omega_1$.
(d) $<_G = \bigcup\{<_p \mid p \in G\}$ is a tree ordering on ω_1.
(e) The tree

$$T_G = (\omega_1, <_G)$$

is an ω_1-tree.

(f) Define a function f_G by

$$f_G(\alpha) = \bigcup\{f_p(\alpha) \mid p \in G \ \wedge\ \alpha \in \mathrm{dom}(f_p)\}$$

Prove that $f_G : \omega_2^{\mathbf{V}} \to B(T_G)$ is an injective function where $B(T_G)$ is the set of all cofinal branches through T_G.

Conclude that if $\mathbf{V} \models \mathsf{CH}$ then in $\mathbf{V}[G]$, the tree T_G is a Kurepa tree.

2. (1 page) Work in ZFC. This forcing is an example of adding a Suslin tree via finite conditions. Define a poset $\mathbb{P}$ as follows.

- **Conditions** are finite trees $p = (u_p, <_p)$ where $u_p \in [\omega_1]^{<\omega}$ satisfying the following condition

$$t <_p t' \implies t < t' \tag{1}$$

  where the ordering on the right side is the natural ordering of ordinals.
- **Ordering** $p \leq q$ iff
  - $u_p \supseteq u_q$ and
  - $<_q = <_p \cap (u_q \times u_q)$.

Prove the following.

(a) $\mathbb{P}$ is c.c.c.

Assume G is $(\mathbb{P}, \mathbf{V})$-generic.

(b) $\bigcup\{u_p \mid p \in G\} = \omega_1$.

(c) The binary relation

$$<_G = \bigcup\{<_p \mid p \in G\},$$

is a tree ordering on ω_1 such that

$$t <_G t' \implies t < t'$$

whenever $t, t' \in \omega_1$.

Let

$$T_G = (\omega_1, <_G)$$

(d) T_G is a tree of height ω_1 such that every node in T_G is a splitting node.

In order to prove that T_G is a Suslin tree in $\mathbf{V}[G]$, assume $\dot{A}$ is a $\mathbb{P}$-term and $p \in \mathbb{P}$ is a condition such that

$$p \Vdash \text{“}\dot{A} : \omega_1 \to \mathbb{P} \text{ and } \operatorname{rng}(\dot{A}) \text{ is a maximal antichain in } \mathbb{P}\text{”}. \tag{2}$$

For each $\xi < \omega_1$ fix a condition $p_\xi \leq p$ and a node $t_\xi < \omega_1$ such that $t_\xi \in u_{p_\xi}$ and

$$p_\xi \Vdash \check{t}_\xi = \dot{A}(\check{\xi}).$$

(e) Show that there is a cofinal set $X \subseteq \omega_1$ such that

$$\xi < \xi' \implies t_\xi < t_{\xi'}$$

whenever $\xi, \xi' \in X$.

(f) Show that there are $\xi < \xi'$ in X far enough from each other such that there is a common extension $p^* \leq p_\xi, p_{\xi'}$ with $t_\xi <_{p^*} t_{\xi'}$.

Organize steps (a) – (f) into a valid forcing argument.

3. (1 page) Work in ZFC. This is another poset for adding a club to ω_1 with finite conditions. A more elaborate variant of this poset will shoot a club through a given stationary set $S \subseteq \omega_1$. The poset $\mathbb{P}$ is defined as follows.

- **Conditions** are finite strictly increasing functions $p : a \to \omega_1$ such that $a \in [\omega_1]^{<\omega}$ and there is a normal function $f : \omega_1 \to \omega_1$ with $p \subseteq f$.
- **Ordering** is reverse inclusion, so $p \leq q$ iff $p \supseteq q$.

Let G be $(\mathbb{P}, \mathbf{V})$-generic and

$$f_G = \bigcup G.$$

Prove the following.

(a) $\omega_1^{\mathbf{V}[G]} = \omega_1^{\mathbf{V}}$. Here apply the method used in the lecture to prove that the Abraham-Shelah forcing does not collapse ω_1.
(b) $f_G : \omega_1 \to \omega_1$ is a normal function.
(c) $f_G \notin \mathbf{V}$.

4. (1/2 page) Work in ZFC. Prove that the Abraham-Shelah forcing as well as the forcing from Problem 3 above are neither c.c.c. nor ω_1-distributive. Nevertheless, you have seen that these posets do not collapse ω_1.

5. (2/3 page) Work in ZFC. Assume κ is regular, $\mathbb{P}$ is a κ-c.c. poset and $\mathbb{Q}$ is a κ-closed poset. Let G be $(\mathbb{P}, \mathbf{V})$-generic and H be $(\mathbb{Q}, \mathbf{V})$-generic. Prove the following.

(a) $\mathbb{P}$ is κ-c.c. in the sense of $\mathbf{V}[H]$.
(b) $\mathbb{Q}$ is κ-distributive in the sense of $\mathbf{V}[G]$.
(c) Is $\mathbb{Q}$ necessarily κ-closed in the sense of $\mathbf{V}[G]$?

6. (1/2 page) Work in ZFC. Let $\mathbb{P}$ be a poset and $G \times H$ be $(\mathbb{P} \times \mathbb{P}, \mathbf{V})$-generic. Prove that

$$\mathbf{V}[G] \cap \mathbf{V}[H] = \mathbf{V}.$$

7. (2/3 page) Work in ZFC. Recall the notion of the regular open algebra from Fall 2019. Recall also that if $\mathbb{P}$ is a poset then we can view $\mathbb{P}$ as the topological space with domain $\mathbb{P}$ and the topology with topology basis

$$\mathbb{B} = \{p\downarrow \mid p \in \mathbb{P}\}.$$

Then $\mathsf{RO}(\mathbb{P})$ is the complete Boolean algebra consisting of all regular open sets in the topological space $\mathbb{P}$; the Boolean algebra ordering is the inclusion.

(a) Prove that the map

$$e : \mathbb{P} \to (\mathsf{RO}(\mathbb{P}) \setminus \{\varnothing\}, \subseteq)$$

defined by

$$e(p) = \overline{(p\downarrow)}^{\circ}$$

is a dense embedding.

This shows that generic extensions via complete Boolean algebras yield all possible generic extensions by all posets, so doing forcing with complete Boolean algebras is not a loss of generality.

(b) Let $\mathbb{B}$ be a complete Boolean algebra, $\mathbb{Q} = (\mathbb{B} \setminus \{0_{\mathbb{B}}\}, \leq_{\mathbb{B}})$, and G be $(\mathbb{Q}, \mathbf{V})$-generic. Notice that b, c are incompatible in $\mathbb{Q}$ iff $b \wedge c = 0_{\mathbb{B}}$. Let further $A \in \mathbf{V}$ and $X \subseteq A$ be such that $X \in \mathbf{V}[G]$. Show that there is a function $f : A \to \mathbb{B}$ such that $f \in \mathbf{V}$ and

$$X = f^{-1}[G].$$

Thus, if $k : \mathbb{B} \to \{0, 1\}$ is the quotient map associated with quotient $\mathbb{B}/G$ (notice that G is an ultrafilter on $\mathbb{B}$) then $k \circ f : A \to \{0, 1\}$ is the characteristic function of X.

This shows that when doing forcing theory using complete Boolean algebras $\mathbb{B}$, we can restrict our definition of terms to consider only functions f with $\mathrm{dom}(f)$ being a set of terms (in this new sense) and $\mathrm{rng}(f) \subseteq \mathbb{B}$.