9.32 Prop. (Bernstein) If there is a well-ordering of reals then the PSP fails.

Proof. Each perfect set is coded by a real. So if \( C \) is a w.o. of reals of type \( \text{card}(\mathbb{R}) \) we can recursively build a set \( A \subseteq \mathbb{R} \) of size \( \text{card}(\mathbb{R}) \) such that \( A \cap C \neq \emptyset \neq A^c \cap C \) for every perfect \( C \subseteq \mathbb{R} \). Then \( A \) is a counterexample to PSP. \( \Box \)

9.33 Corollary (ZF). If PSP holds then there does not exist any injection \( f : \omega_1 \rightarrow \mathbb{R} \).

Proof. If \( f \) is such an injection then \( \text{rng}(f) \) is an uncountable set of reals, so there is some perfect \( C \subseteq \text{rng}(f) \). But then \( C \) is well-ordinalable. But in ZF one can construct a bijection \( g : \mathbb{R} \rightarrow C \). So \( \mathbb{R} \) would be well-ordinalable. Contradiction with Prop. 9.32. \( \Box \) Cor. 9.33.

10. A GLIMPSE INTO FINE STRUCTURE THEORY

10.1 Theorem. There is an \( \Sigma_1 \)-formula \( \phi(v, v_1) \) which defines \( \Delta_0 \)-satisfaction over any transitive \( \Sigma_1 \)-closed well-founded transitive structure. It follows that \( \Delta_0 \)-satisfaction is a \( \Delta_1 \)-properly transitive \( \Sigma_1 \)-closed structure.

Here \( M \models \phi[v, s] \iff M \models \phi(\check{v}, \check{s}) \)

where \( \check{v} \) is the code of \( v \) under some recursive coding fixed in advance. \( s \) is an evaluation of variables in \( v \).

Proof. \( \Delta_0 \)-satisfaction is \( \Delta_1 \) because
\[ M \models \phi(\check{v}, s) \iff M \models \phi(\check{v}, s) \]

Construction of \( \phi \): Let's abuse the notation and write \( \phi \) for \( \phi^1 \). Point: if \( \phi \) is \( \Delta_0 \) then
\[ M \models \phi[v, s] \iff (u, e, Anu) \subseteq \phi[v,s] \]
where \( u = \mathbf{u}_1 u_1 \cdots u_{n} u_{n} \) and \( m = \) the \# of quantifiers in \( \phi \).

So: \( M \models \phi[s] \iff \exists \exists u \exists v \exists w \exists \exists x \exists y \exists z \exists d \)

(a) \( m, u, v \) are finite ordinals \( m \)
(b) \( u = \mathbf{u}_1 u_1 \cdots u_1 u_{n} u_{n} \) and \( d = \mathbf{d}_1 d_1 \cdots d_1 d_{n} d_{n} \)
(c) \( f : u + 1 \to \text{Formulas} \) and \( g : (u + 1) \times u \to \{ 0, 1 \} \)
(d) \( \forall i < u + 1 \) \( \exists v, u', d \)

\[ f(i) = \begin{cases} 0 & \text{if } u = u' \land u = u' \land f(i) = \begin{cases} 1 & \text{if } v \in A \\ 0 & \text{if } \exists i < i \land f(i) = \begin{cases} 1 & \text{if } j \in \mathbf{u}_1 \land i \in \mathbf{u}_1 \\ 0 & \text{if } i \end{cases} \\ 0 & \text{if } \exists i < i \land f(i) = \begin{cases} 1 & \text{if } (\exists v \in u') f(i) = \begin{cases} 1 & \text{if } j \in \mathbf{u}_1 \land i \in \mathbf{u}_1 \\ 0 & \text{if } i \end{cases} \\ 0 & \text{if } \end{cases} \end{cases} \]

(2) \( \phi = f(u) \)

(3) \( \forall i < u + 1 \forall v \in d \)

\[ f(i) = \begin{cases} 1 & \text{if } v = v' \land f(i) = \begin{cases} 1 & \text{if } s(w) = s(v') \\ 0 & \text{if } s(w) \notin A \\ 0 & \text{if } s(w) \notin A \\ 0 & \text{if } i \end{cases} \\ 0 & \text{if } \exists i < i \land f(i) = \begin{cases} 1 & \text{if } j \in \mathbf{u}_1 \land i \in \mathbf{u}_1 \\ 0 & \text{if } i \end{cases} \\ 0 & \text{if } \exists i < i \land f(i) = \begin{cases} 1 & \text{if } (\exists v \in u') f(i) = \begin{cases} 1 & \text{if } j \in \mathbf{u}_1 \land i \in \mathbf{u}_1 \\ 0 & \text{if } i \end{cases} \\ 0 & \text{if } (\exists v \in u') f(i) = \begin{cases} 1 & \text{if } j \in \mathbf{u}_1 \land i \in \mathbf{u}_1 \\ 0 & \text{if } i \end{cases} \\ 0 & \text{if } \end{cases} \end{cases} \]

(3) \( g(u, s) = 1 \).

Add (2) to (b): (3) "if \( u \) is a variable occurring in \( \phi \) then \( u \in d \)."

**Thm 10.7**

10.2. Corollary. There is a \( \Sigma_1 \) formula \( \phi^* \) such that for all \( \Sigma_1 \) formulas \( \phi \) and all evaluating \( s \) of \( \phi^* \) and all transitive \( A \)-sets \( \mathcal{M} \):

\( M \models \phi^*[s] \iff M \models \phi^*[s] \).

Proof (sketch). For \( \Sigma_1 \) : if \( \phi \) is \( \Sigma_1 \) then \( \phi = \exists \theta \psi \theta \).

For some \( \Delta_0 \) formula \( \theta \), then
\[ M \models \varphi(s) \iff (\exists a \in M) M \models \varphi(\langle a, s \rangle)
\]
\[ \frac{M \models \exists a \phi(\langle a, s \rangle)}{M \models \exists a \phi(\langle a, s \rangle)}
\]

And note the translation between \( \varphi \) and \( \chi \) is reciprocal.

**Note:** If \( \varphi \equiv \Pi_1 \), we end up with \( (\exists a) \phi(\langle a, s \rangle) \) but this can be replaced with \( (\forall \alpha) \phi(\langle \alpha, s \rangle) \) which is \( \Pi_1 \).

[Ca 10.2.]

**10.3 Prop.** Assume \( M \) is a transitive \( \mathcal{L} \)-closed structure with a \( \mathcal{L} \)-definable well-order \( < \). Let \( X \subseteq M \)

\( \gamma \) the set of all \( y \in M \) which are \( \Sigma_{n+1}(M) \)
definable from parameters in \( X \).

Then \( X \subseteq Y \) and \( \gamma \subseteq \Sigma_n M \).

**Proof.** That \( X \subseteq Y \) is easy. To see \( \gamma \subseteq \Sigma_n M \):

Let \( \alpha(v_0, v_1, \ldots, v_n) \) be a \( \Sigma_n \)-formula and \( a, \ldots, a_n \in Y \) and

assume \( M \models (\exists v_0) \alpha(v_0, a, \ldots, a_n) \). To each \( i \in \{0, \ldots, k\} \)
pick a \( \Sigma_{n+1} \)-formula \( \varphi_i(v_0, v_1, \ldots, v_n) \) and \( x_0, \ldots, x_k \in X \) s.t.

\[ \varphi_i = \text{the unique } \exists \text{ formula } M \models \varphi_i(x_0, \ldots, x_k) \]

By introducing during variables \( w, u \) a.a. \( \exists w, x_0, \ldots, x_n \) are the same for all \( i \). Then we can find \( b \in Y \) s.t.

\[ M \models \varphi(b, a, \ldots, a_n) \] by minimality,

\[ v_0 = b \models (\exists v_0) \ldots (\exists v_k) \left[ \bigwedge_{i=0}^{n+1} \varphi_i(v_0, v_1, \ldots, v_n) \land \varphi(v_0, v_1, \ldots, v_k) \land \left( \forall v_0 \right) \left( v_0 < v_0 \rightarrow \neg \varphi(v_0, v_1, \ldots, v_k) \right) \right] \]

This statement has the form \( (\exists v_0) \ldots (\exists v_k) \Pi_n \)
so is \( \Sigma_{n+1} \), and defines \( b \) from parameters

in \( X \). So \( b \in \gamma \).