MATH 281C L1 2020-03-31

9.32 Prop (ZF) (Bernstein) If there is a well-ordering of reals then the PSP fails.

Proof Each perfect set is coded by a real. So if $<$ is a w.o. of reals of type card($\mathbb{R}$) we can recursively build a set $A \subseteq \mathbb{R}$ of size card($\mathbb{R}$) such that $A \cap C \neq \emptyset \neq A^c \cap C$ for every perfect $C \subseteq \mathbb{R}$. Then A is a counterexample to PSP. $\boxtimes$

9.33. Corollary (ZF) If PSP holds then there does not exist any injection $f: \omega_1 \to \mathbb{R}$

Proof If f is such an injection then $\mathrm{rng}(f)$ is an uncountable set of reals, so there is some perfect $C \subseteq \mathrm{rng}(f)$. But then C is well-ordrable. But in ZF one can construct a bijection $g: \mathbb{R} \to C$. So $\mathbb{R}$ would be well-ordrable. Contradiction with P 9.32. $\boxtimes$ Cor 9.33.

10. A GLIMPSE INTO FINE STRUCTURE THEORY

10.1. Theorem There is a Σ_1-formula $\Phi(v_0, v_1)$ which defines Δ_0-satisfaction over any transitive A-rudimentarily closed LST_A-structures. It follows that Δ_0-satisfaction is a Δ_1-property over transitive A-rud. closed structures.

Here $M \models \varphi[s] \iff M \models \Phi(\ulcorner\varphi\urcorner, s)$

where $\ulcorner\varphi\urcorner$ is the code of φ under some recursive coding fixed in advance. s is an evaluation of variables in φ.

Proof Δ_0-satisfaction is Δ_1 because

$$M \not\models \Phi(\ulcorner\varphi\urcorner, s) \quad \text{iff} \quad M \models \underbrace{\Phi(\ulcorner\neg\varphi\urcorner, s)}_{\Sigma_1}$$

Construction of Φ: Let's abuse the notation and write φ for $\ulcorner\varphi\urcorner$. Point: if φ is Δ_0 then

$$M \models \varphi[s] \quad \text{iff} \quad (u, \in, A \cap u) \models \varphi[s]$$

where $u = \mathrm{rng}(s) \cup \bigcup \mathrm{rng}(s) \cup \cdots \cup \bigcup^m \mathrm{rng}(s)$
and m = the # of quantifiers in φ

So: $M \models \varphi[s]$ iff

$$(\exists m)(\exists n)(\exists u)(\exists f)(\exists g)(\exists d)$$

(a) m, n are finite ordinals

(b) $u = \mathrm{rng}(s) \cup \bigcup \mathrm{rng}(s) \cup \cdots \cup \bigcup^m \mathrm{rng}(s) \wedge d = \mathrm{dom}(s) \wedge$ ⊛

(c) $f : n+1 \longrightarrow \text{Formulae} \wedge g : (n+1) \times {}^d u \longrightarrow \{0,1\}$

(d) $(\forall i < n+1)(\exists v, v' \in d)$

$\quad [\, f(i) = \text{“} v = v' \text{''} \vee f(i) = \text{“} v \in v' \text{''} \vee f(i) = \text{“} v \in A \text{''} \vee$

$\quad \vee (\exists j < i)(f(i) = \neg f(j)) \vee (\exists j_1, j_2 < i)(f(i) = f(j_1) \wedge f(j_2))$

$\quad \vee (\exists j < i)(f(i) = (\exists v \in v') f(j))$

(e) $\varphi = f(n)$

(f) $(\forall i < n+1)(\forall s \in {}^d u)$

- $f(i) = \text{“} v = v' \text{''} \Rightarrow g(i,s) = 1$ iff $s(v) = s(v')$
- $f(i) = \text{“} v \in v' \text{''} \Rightarrow g(i,s) = 1$ iff $s(v) \in s(v')$
- $f(i) = \text{“} v \in A \text{''} \Rightarrow g(i,s) = 1$ iff $s(v) \in A$
- $(\exists j < i)\, f(i) = \neg f(j) \Rightarrow g(i,s) = 1 \wedge g(j,s)$
- $(\exists j_1, j_2 < i)(f(i) = f(j_1) \wedge f(j_2)) \Rightarrow g(i,s) = g(j_1,s) \wedge g(j_2,s)$
- $(\exists j < i)(\exists v, v' \in d)(f(i) = (\exists v \in v') f(j)) \Rightarrow$

$$g(i,s) = \bigcup \{ g(j,s') \mid s' \in d \wedge s'(v^*) = s(v^*) \wedge s'(v) \in s'(v') \text{ for all } v^* \neq v,\ v^* \in d \}$$

(g) $g(n,s) = 1$.

Add ⊛ to (b): ⊛ “If v is a variable occurring in φ then $v \in d$”

☒ Thm 10.1

10.2. Corollary There is a Σ_n / Π_n formula $\Phi_n(v_0, v_1)$ in LST_A such that for all Σ_n / Π_n-formulae $\varphi(\vec{w})$ and all evaluations s of $\vec{w}$ and all transitive A-rud. closed M:

$$M \models \varphi(s) \text{ iff } M \models \Phi_n(\varphi, s)$$

Proof (Sketch) For Σ_1: if φ is Σ_1 then $\varphi \equiv (\exists v)\psi(v, \vec{w})$ for some Δ_0-formula ψ. Then

$M \models \varphi(s)$ iff $(\exists a \in M)\, M \models \psi(\langle a\rangle^\frown s)$

iff $M \models (\exists a)\, \phi(\psi, \langle a\rangle^\frown s)$

And note the translation between φ and ψ is recursive.

Note: If φ is Π_1 we end up with $(\forall a)\, \phi(\psi, \langle a\rangle^\frown s)$ but this can be replaced with $(\forall a) \neg \phi(\neg\psi, \langle a\rangle^\frown s)$ which is Π_1. $\square$ Cor 10.2.

10.3. Prop Assume M is a transitive A-rud closed structure with a Σ_1-definable well-ordering $<$. Let $X \subseteq M$ and

Y = the set of all $y \in M$ which are $\Sigma_{n+1}(M)$ definable from parameters in X

Then $X \subseteq Y$ and $Y \prec_{\Sigma_n} M$.

Proof That $X \subseteq Y$ is easy. To see $Y \prec_{\Sigma_n} M$: Let $\varphi(v_0, v_1 \ldots v_k)$ be a Σ_n-formula and $a_1 \ldots a_k \in Y$ and assume $M \models (\exists v_0)\, \varphi(v_0, a_1 \ldots a_k)$. To each $i \in \{1 \ldots k\}$ pick a Σ_{n+1}-formula $\psi_i(v_i, u_1 \ldots u_\ell)$ and $x_1 \ldots x_\ell \in X$ s.t.

a_i = the unique z s.t. $M \models \psi_i(z, x_1 \ldots x_\ell)$

By introducing dummy variables w.m.a. that $\ell, x_1 \ldots x_\ell$ are the same for all i. Then we can find $b \in Y$ s.t. $M \models \varphi(b, a_1 \ldots a_k)$ by minimality:

$$v_0 = b \iff (\exists v_1) \cdots (\exists v_k) \Big[\underbrace{\bigwedge_{i=1}^{k} \psi_i(v_i, x_1 \ldots x_\ell)}_{\Sigma_{n+1}} \wedge \underbrace{\varphi(v_0, v_1 \ldots v_k)}_{\Sigma_n} \wedge$$

$$\underbrace{(\forall v_0')(\underbrace{v_0' < v_0}_{\Sigma_1} \to \underbrace{\neg \varphi(v_0', v_1 \ldots v_k)}_{\Pi_n})}_{\Pi_n} \Big]$$

This statement has the form $(\exists v_1) \ldots (\exists v_k)\, \Pi_n$ so is Σ_{n+1}, and defines b from parameters in X. So $b \in Y$. $\square$ P. 10.3.