MATH 281C SPRING 2020 L2

10.4. Def Given a binary relation R we say that a partial function f uniformizes R iff
(a) For every x: if the section $R_x = \{y \mid xR_y y\}$ is nonempty then $x \in \mathrm{dom}(f)$
(b) For every x as in (a): $(x, f(x)) \in R$

10.5. Prop Assume M is a transitive A-md. closed structure with a Σ_1-definable well-ordering $<$ and $R \subseteq M$ is a binary relation which is $\Sigma_n(M)$. Then R has a uniformization which is $\Sigma_{n+1}(M)$.
Moreover, such a uniformization can be obtained uniformly, i.e. its definition depends only on the definition of R and the w.o. $<$.
Proof like the proof of Prop 10.3. $\boxtimes$ Prop 10.5.

10.6. Prop Assume $f: A \to B$ is a function which is Σ_n-definable and total on A. Then f is Δ_n-definable.
Proof $y \neq f(x) \iff (\exists y')(y' = f(x) \wedge y' \neq y)$
Point: We do not need to check that $x \in \mathrm{dom}(f)$ $\boxtimes$ 10.6.

10.7. Proposition Assume M is a transitive A-md. closed structure with a Σ_1-definable well-ordering $<$. Then M has definable Skolem functions, more precisely for each $n \in \omega$ there is a partial function $h_n^M(x,y)$ s.t.
(a) h_n^M is $\Sigma_{n+1}(M)$-definable
(b) For each Σ_n-formula $\varphi(v_0, v_1, \dots v_k)$ and each tuple $a_1, \dots a_k$:
if there is $b \in M$ s.t. $M \models \varphi(b, a_1 \dots a_k)$ then
$h_n^M(\ulcorner\varphi\urcorner, \langle a_1 \dots a_k\rangle)$ is defined and
$$M \models \varphi(h_n^M(\ulcorner\varphi\urcorner, \langle a_1 \dots a_k\rangle), a_1 \dots a_k)$$

When talking about codes $\ulcorner\varphi\urcorner$, we think of some recursive enumeration of Σ_n-formulae $(\varphi_i \mid i \in \omega)$ fixed in advance, and and think of i as the code of φ_i. We would write "$h_n^M(i, \langle a_1 \dots a_k\rangle)$" in (b) above.
These Skolem functions are often called crude Skolem function because of their higher complexity.

Proof Let Φ_n be the definition of the Σ_n-satisfaction relation from Cor 10.2. Then uniformize the relation
$R(x,y) \equiv \Phi_n((x)_0, \langle y\rangle^\frown x^*)$ using Prop 10.5.
Here we think of x a sequence $\langle i, a_1 \dots a_k\rangle$ as in Prop 10.7.
x^* is the sequence obtained by removing the first member;
$(x)_0$ is the first member. $\boxtimes$ Prop 10.7.

10.8. Proposition Assume $\tau < \omega(\alpha+1)$ and $f \in J_{\alpha+1}$ is s.t. $f : \tau \xrightarrow{\text{onto}} J_\alpha$.
Then there is $a \subseteq \tau$ s.t. $a \in J_{\alpha+1} \setminus J_\alpha$.
(This is also true for the relativized hierarchy J^A_α.)

Proof: Cantor diagonalization:

$$\xi \in a \iff \xi \notin f(\xi)$$

$\boxtimes$ P 10.8.

10.9. Def The structure J^A_β is **acceptable** iff for every $\alpha < \beta$ and every $\tau < \alpha$ which is a cardinal in J^A_α:
if there is some $a \subseteq \tau$ s.t. $a \in J_{\alpha+1} \setminus J_\alpha$ then there is a surjection $f : \tau \xrightarrow{\text{onto}} J_\alpha$ s.t. $f \in J_{\alpha+1}$.

10.10. Remark Acceptability is a strong form of GCH.

10.11. Theorem Each structure J_β is acceptable.

Proof Assume $\tau < \alpha < \beta$ and $a \in J_{\alpha+1} \setminus J_\alpha$ s.t. $a \subseteq \tau$, τ cardinal in J_α.
By T 8.21, a is definable over J_α. So there is some $n \in \omega$, some Σ_n-formula $\varphi(u,v)$ and some $p \in J_\alpha$ s.t. for all $\xi < \tau$:

$$\xi \in a \iff J_\alpha \models \varphi(\xi, p)$$

Let $Y = \{ h_n^{J_\alpha}(i, \langle \xi_1 \ldots \xi_k, p \rangle) \mid i \in \omega \wedge \xi_1 \ldots \xi_k < \tau \}$

Then $Y \prec_{\Sigma_n} J_\alpha$ by Prop 10.3.

Let $J_{\bar\alpha}$ be the transitive collapse of Y and $\sigma : J_{\bar\alpha} \to J_\alpha$ be the inverse of the collapsing map. Then

(1) σ is Σ_n-preserving

(2) $\sigma \restriction \tau = \mathrm{id} \restriction \tau$

(3) Letting $\bar p \in J_{\bar\alpha}$ such that $\sigma(\bar p) = p$ (Notice $p \in Y$):

$$a = \{ \xi < \tau \mid J_{\bar\alpha} \models \varphi(\xi, \bar p) \}$$

So $a \in J_{\bar\alpha + 1}$

(4) $\bar\alpha = \alpha$

To see (4): $\bar\alpha \leq \alpha$ by the existence of σ
$\bar\alpha \not< \alpha$ because $a \in J_{\bar\alpha+1} \setminus J_\alpha$

Now notice the map

$$(i, x) \longmapsto h_n^{J_\alpha}(i, x^\frown \langle \bar p \rangle)$$

is a partial surjection of $\omega \times {}^{<\omega}\tau$ onto J_α, which is Σ_{n+1}-definable over J_α. This surjection is an element of $J_{\alpha+1}$. This surjection can be turned into a surjection $f : \tau \to J_\alpha$ s.t. $f \in J_{\alpha+1}$.

$\boxtimes$ Thm 10.11.

10.12. Remark (Discussed previously)

LO = the set of all $a \in \mathcal{N}$ which code linear orderings on ω; the linear ordering coded by a is

$<_a = \{ \langle i,j \rangle \in \omega \mid a(\langle i,j \rangle) = 0 \}$

WO = the set of all $a \in$ LO s.t. $\underline{a}$ is a well-ordering.

LO is an arithmetic subset of $\mathcal{N}$

WO is a Π^1_1-subset of $\mathcal{N}$

There is a universal Σ^1_1-set U: This means:

- $U \subseteq \mathcal{N} \times \mathcal{N}$ is Σ^1_1
- If $A \subseteq \mathcal{N}$ is $\underset{\sim}{\Sigma}^1_1$ then there is some $a \in \mathcal{N}$ s.t. for all $x \in \mathcal{N}$

$$x \in A \iff (a,x) \in U$$

From this, we get universal sets for any Σ^1_n, Π^1_n.

The universal Σ^1_n / Π^1_n-set is $\underline{\text{not}}$ $\underset{\sim}{\Pi}^1_n / \underset{\sim}{\Sigma}^1_n$.