M281C SPRING 2020 L3

Recall: There is a Π^1_1 set $K \subseteq \mathcal{N}$ which is not $\underset{\sim}{\Sigma}^1_1$.

Why: Let $U \subseteq \mathcal{N}\times\mathcal{N}$ be universal Σ^1_1. (So U is lightface Σ^1_1). Let

$$x \in K \iff \langle x, x\rangle \notin U$$

Then K is Π^1_1. If K were $\underset{\sim}{\Sigma}^1_1$ then there would be some $a \in \mathcal{N}$ s.t. $x \in K \iff \langle a, x\rangle \in U$ for all $x \in \mathcal{N}$. For $x = a$ we get

$$\langle a, a\rangle \in U \iff a \in K \iff \langle a, a\rangle \notin U.$$

Contradiction.

Recall again: WO is a Π^1_1-subset of $\mathcal{N}$.

10.13 Theorem (Σ^1_1-boundedness Thm). Assume $A \subseteq \mathrm{WO}$ and A is $\underset{\sim}{\Sigma}^1_1$. Then there is some $\delta < \omega_1$ s.t.

$$\{\mathrm{otp}(<_a) \mid a \in A\} \subseteq \delta,$$

i.e. the set of all $\mathrm{otp}(<_a)$ for $a \in A$ is bounded below ω_1.

In particular, if $C \subseteq \mathrm{WO}$ is a perfect set then the set of all $\mathrm{otp}(<_a)$ for $a \in C$ is bounded below ω_1.

Proof We know there is some recursive T on $\omega\times\omega$ such that for every $x \in \mathcal{N}$: $x \in K \iff T_x$ is well-founded $\iff <^{T_x}_{BK}$ is a w.o.

If $A \subseteq \mathrm{WO}$ is s.t. $\mathrm{otp}(<_a)$ for $a \in A$ is unbounded in ω_1 then

$$(1)\quad x \in K \iff (\exists^{\mathcal{N}} a)(\exists^{\mathcal{N}} f)\big(\underbrace{a \in A}_{\Sigma^1_1} \wedge \underbrace{f : (\omega, <^{T_x}_{BK}) \xrightarrow{\text{o.p.}} (\omega, a)}_{\text{Arithmetic.}}\big)$$

Here "o.p." means "order preserving".

So the RHS in (1) is $\underset{\sim}{\Sigma}^1_1$. Contradiction, as K is not $\underset{\sim}{\Sigma}^1_1$. $\square$ T10.13.

10.14. Proposition Assume $A \subseteq \mathrm{WO}$ s.t.

(a) $\{\mathrm{otp}(<_a) \mid a \in A\}$ is unbounded in ω_1

(b) For every $\delta < \omega_1$ the set $\{a \in A \mid \mathrm{otp}(<_a) < \delta\}$ is countable.

Then A is a counterexample to PSP.

Proof: Use T 10.13.

10.15 Theorem In L there is a Π^1_1-set A as in P 10.14. More generally, if $a \in \mathcal{N}$ then in $L[a]$ there is a $\Pi^1_1(a)$-set A as in P 10.14.

Proof For L: We define A as follows:

(1) $\quad a \in A \iff a \in WO \wedge a \in J_{otp(<_a)+1}$

First check that A satisfies (a) and (b) in P 10.14. The verification of (b) is easy (Exercise). To see (a): Fix $\alpha < \omega_1$, let

$$Y = \{ h_1^{J_{\omega_1}}(i, s^\frown\langle\alpha\rangle) \mid s \in {}^{<\omega}\omega \}$$

Then $Y \prec_{\Sigma_1} J_{\omega_1}$ and $\omega \subseteq Y$ so Y is transitive, hence $Y = J_\beta$ for some $\beta < \omega_1$. The Skolem function induces a total definable surjection $f : \omega \xrightarrow{onto} \omega\beta$. Consider the set $Z \subseteq \omega$ defined by:

$$n \in Z \iff f(n) \in \omega\beta \wedge (\forall k < n)(f(k) \neq f(n))$$

Then $f \restriction Z$ is an injection, and the binary relation $<_Z$ on Z defined by $n <_Z n' \iff f(n) \in f(n')$ is a w.o. on Z of order type $\omega\beta$. Then notice that there is a definable (over J_β) bijection $g : \omega \to Z$; using this bijection we can get a definable well-ordering (over J_β) on ω of order type $\omega\beta$. Then get an element $a \in \mathcal{N}$ coding this w.o. in the usual way. Then a is also definable over J_β, so $a \in J_{\beta+1}$. This verifies (a) in P10.14.

Now we need to see that the RHS in (1) is Π^1_1. We express it in this way:

(2) $\quad (\forall M)(M = J_{otp(<_a)+1} \to a \in M)$

We can do this as we know that M exists for "many" $a \in WO$; "many" means that $otp(<_a)$ are unbounded in ω_1.

Some technical details:

- let σ be a Q-sentence in LST expressing "I am a J"
- $\sigma_i(v)$ be a formula in LST with single free variable v expressing "v is the natural number i"

We will think of $b \in \mathcal{N}$ as realizing the symbol $\dot{\in}$. Then (2) can be rewritten as

(3) $$(\forall^{\mathcal{N}} b)\Big[\big(\underbrace{Sat(b,\sigma,\emptyset)}_{\substack{\Delta^1_1 \\ \text{I am } J_*}} \wedge \underbrace{\varphi_1(b,a)}_{\Sigma^1_1}\big) \to \underbrace{\varphi_2(b,a)}_{\Delta^1_1}\Big]$$

$\varphi_1(b,a)$ will say that b codes the structure $J_{otp(<_a)+1}$

Let
$\rho(v)$ be a LST formula with single free variable v expressing "v is an ordinal"
$\rho_s(v)$ be a LST formula with single free variable v expressing "v is a successor ordinal"

Then

$\varphi_1(b,a) \equiv (\exists^{\omega} n)(\exists^{\omega^\omega} f)$
(i) $Sat(b, \rho, v \mapsto n)$
(ii) $(\forall k)\big[\big(b(\langle n,k\rangle = 0) \wedge Sat(b, \rho, v \mapsto k)\big)\big] \to Sat(b, \rho_s, v \mapsto k)$
(iii) $f : (\omega, <_a) \to (\{x \in M \mid b(\langle x, n\rangle) = 0\}, <_b)$
is a bijection

(i) + (ii) are Δ^1_1, (iii) is arithmetic. (Exercise)
(i) + (ii) say that the model coded by b has only successor ordinals above n.
(iii) Says that the ordinals below n in the sense of the model coded by b are isomorphic to $<_a$.

The formula $\varphi_2(b,a)$ expresses that the model coded by b has its version of a, call it m, in the following sense:
For every $i,j \in \omega$: $\langle i,j\rangle \in a \iff$ the model coded by b believes that its version of $\langle i,j\rangle$ is in m.

In other words: If we transitive collapse the model coded by b then the the transitive collapse of m is a.

For $i,j \in \omega$ let $\varphi_{ij}(v)$ be the formula
$\varphi_{ij}(v) \equiv (\exists z)(\exists u)(\exists u')(z = \langle u,u'\rangle \wedge \sigma_i(u) \wedge \sigma_j(u') \wedge z \in v)$
So $\varphi_{ij}(v)$ expresses "the ordered pair representing $\langle i,j\rangle$ is in v"
In the sense of the model coded by b.
Then we let
$\varphi_2(b,a) \equiv (\exists m)(\forall i)(\forall j)(\langle i,j\rangle \in x \leftrightarrow Sat(b, \varphi_{ij}, v \mapsto m))$
Point is the assignment $\langle i,j\rangle \mapsto \varphi_{ij}$ is recursive.

So our formula in (3) has the form

$$\underbrace{\forall r \, \Big[\underbrace{\big(\underbrace{\Delta^1_1 \wedge \Sigma^1_1}_{\Sigma^1_1} \big) \longrightarrow \Delta^1_1}_{\Pi^1_1} \Big]}_{\Pi^1_1}$$

$\boxtimes$ T 10.15.