MATH 281C SPRING 2020 L4

Last time: We proved that in $L[a]$ ($a \in \mathcal{N}$) there is a $\Pi^1_1(a)$ subset of WO which satisfies the assumptions of P 10.14. So assuming $V = L[a]$, such a set has no perfect subset.

10.16. Corollary The statement "Every Π^1_1 subset of $\mathcal{N}$ has a perfect subset" is not a theorem of ZFC.

But this gives us more: By Π^1_1-absoluteness, the $\Pi^1_1(a)$ set A constructed in T 10.15 is $\Pi^1_1(a)$ in the sense of V!

Now assume ω_1^V is a cardinal successor in L. So in L there is a cardinal κ such that $\omega_1^V = \kappa^{+L}$. But κ is countable in the sense of V, so working in V: there is some $\underline{a} \in WO$ such that $\underline{a}$ codes the order-type κ. This means:

$$\omega_1^{L[a]} = \omega_1^V$$

Then the set A satisfies the assumptions of P. 10.14 in the sense of V, so A is a $\Pi^1_1(a)$ set which is a counterexample to PSP in the sense of V! So we have proved:

10.17. Theorem If every $\underset{\sim}{\Pi}^1_1$-set has the PSP then ω_1^V is a limit cardinal in L. (More generally, in $L[x]$ for every $x \in \mathcal{N}$.)

10.18. Corollary Assume ω_1^V is regular. Then

$$\mathrm{PSP}(\underset{\sim}{\Pi}^1_1) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \exists \text{ an inaccessible})$$

that is, for instance it is not possible to prove

$$\mathrm{Con}(\mathrm{ZFC}) \Rightarrow \mathrm{Con}(\mathrm{ZF} + \mathrm{PSP}(\underset{\sim}{\Pi}^1_1) + \omega_1^V \text{ is regular})$$

10.19. Remark It is possible to prove

$$\mathrm{Con}(\mathrm{ZFC}) \Rightarrow \mathrm{Con}(\mathrm{ZF} + \mathrm{PSP}(\underset{\sim}{\Pi}^1_1) + \omega_1^V \text{ is singular})$$

(Theorem of J. Truss)

MORE ON FINE STRUCTURE

10.20 THEOREM (Σ_1-uniformization) Structures (J^A_α, B) admit uniform Σ_1-uniformization, i.e. If $\varphi(u,v)$ is a Σ_1-formula in $LST_{A,B}$ then there is Σ_1 formula $\psi(u,v)$ such that for every $\alpha \in On$,

$\psi(u,v)$ defines a function which uniformizes the relation $R_\varphi = \{\langle u,v\rangle \mid M \models \varphi(u,v)\}$ when $M = (J_\alpha^A, B)$.

The ψ is obtained from φ in a recursive way.

<u>Proof</u> Say $\varphi(u,v) \equiv (\exists w)\,\varphi_0(u,v,w)$ where φ_0 is Δ_0. We then let $\tilde\psi(u,v,w) \equiv \langle v,w\rangle$ is the $<_A$-least pair $\langle v',w'\rangle$ s.t. $\varphi_0(u,v',w')$. Then $\tilde\psi$ defines a function (partial) $\tilde f(u)$ with output $\langle v,w\rangle$. If $l(z)$ is the rudimentary function "the left component" then $l\circ\tilde f$ is the uniformizing function for R_φ and it is easy to write its definition using definitions of l, $\tilde f$. Now we show $\tilde f$ has a Σ_1 definition which only depends on φ_0.

$\tilde\psi(u,v,w) \equiv (\exists z)(\exists S)(\exists W)$

- $\underbrace{\mathrm{On}(z)}_{\Delta_0} \wedge \underbrace{S = S_z^A}_{\Sigma_1} \wedge \underbrace{W = W_z^A}_{\Sigma_1}$

$$\Delta_0 \left\{ \begin{array}{l} \bullet\ \langle v,w\rangle \in S \\ \bullet\ \varphi_0(u,v,w) \\ \bullet\ (\forall v',w' \in S)\,(\underbrace{\langle\langle v',w'\rangle,\langle v,w\rangle\rangle \in W}_{\langle v',w'\rangle <_A \langle v,w\rangle} \to \underbrace{\neg\varphi_0(u,v',w')}_{\Delta_0}) \end{array}\right.$$

This proves $\tilde\psi$ is Σ_1. $\square$ T 10.20.

<u>10.21. Remark</u> The task for the Fine Structure Theory is to do this for complexities higher than Σ_1. Point of T 10.20 would be false for Σ_n when $n>1$. Lévy hierarchy does not work here, even in L, as shown by Jensen. One way is to replace Lévy hierarchy with a different one. Jensen: Σ^*-hierarchy, Steel: r-Σ_n.

<u>Before going further:</u> Switch to formulae with a single free variable v. Let $\langle \varphi_i(v) \mid i\in\omega\rangle$ be a fixed recursive enumeration of such formulae. We let

$$\phi_1^A(i,x) \equiv \phi_1(\ulcorner\varphi_i\urcorner, v \mapsto x)$$

We can go back to m arguments by considering $\langle x_1 \ldots x_m\rangle$ as x. Using T 10.20 we can Σ_1-uniformize the relation with definition

$\phi_1^A(i, \langle u,v\rangle)$ (u is argument, v is the output)

this will give us a partial function $h_M(i,u)$ such that for every structure $M = \langle J_\alpha^A, B\rangle$:

- h_M has the same Σ_1-definition, which only depends on Φ_1^1
- h_M is a Σ_1-Skolem function for $\varphi_1^1(i, \langle u, v \rangle)$, and hence a universal Σ_1-Skolem fcn.

10.22. Corollary Assume $M = \langle J_\alpha^A, B \rangle$ and $X \subseteq M$. Let

$$Y = \{ h_M(i, s) \mid s \in {}^{<\omega}X \} \overset{\text{def}}{=} h_M(X)$$

Then $Y \prec_{\Sigma_1} M$. (More precisely: $\langle Y, \in, B \cap Y \rangle \prec_{\Sigma_1} M$)

Proof Exercise, like the proof of P 10.3. But here we can use the properties of h_M. $\square$ 10.22.

10.23. Def Let $M = (J_\alpha^A, B)$ be acceptable (This means: J_α^A is acceptable, as B does not contribute to the hierarchy J_α^A) and amenable, i.e. $B \cap x \in M$ all $x \in M$. The first projectum ρ_M^1 of M is the smallest ordinal ρ s.t. there exists a $\underset{\sim}{\Sigma}_1(M)$ set A such that $A \cap \omega\rho \notin M$. (Note: $\rho_M^1 \geq \omega$)

10.24. Def Let M be as in D.10.23. The set P_M^1 is defined as follows:

$$p \in P_M^1 \iff p \in [On \cap M]^{<\omega} \wedge \text{There is a } \underset{\sim}{\Sigma}_1(M) \text{ set } A \text{ s.t. } A \cap \rho_M^1 \notin M$$

The elements of P_M^1 are called good parameters.

10.25. Proposition $h_M(On \cap M) = M$. So if A is $\Sigma_1(M)$ in some parameter $q \in M$ then A is $\Sigma_1(M)$ in some $p \in P_M^1$; it suffices to take p so that $q = h_M(i, p)$.

Proof Exercise.

10.26 Def Again M is as in D 10.23. The set R_M^1 is defined as follows:

$$p \in R_M^1 \iff p \in [On \cap M]^{<\omega} \text{ and } h_M(\rho_M^1 \cup \{p\}) = M$$

10.27. Prop M as above

(a) $P_M^1 \neq \emptyset$

(b) $R_M^1 \subseteq P_M^1$

Proof (a) Trivial (b) This is done by the diagonalization argument from P.10.8. One just need to check that if f in that argument is $\Sigma_1(M)$ the diagonal set is also $\Sigma_1(M)$. □ P10.27.

10.29 Remark Recall we have the set-like well-ordering on $[On]^{<\omega}$ defined by:

$$p <^* q \iff \text{there is } \alpha \in q \setminus p \text{ s.t. } p \setminus (\alpha+1) = q \setminus (\alpha+1)$$

10.30. Def The first standard parameter of M is defined as follows:

$$p^1_M = \text{the } <^*\text{-least element of } P^1_M$$