Last time: We proved that in \( L[a] \) (at \( a \in V \)) there is a \( \Pi^1_1(a) \) subset of \( W \) which satisfies the assumptions of 10.14. So assuming \( V = L[a] \), such a set has no perfect subset.

10.16. Corollary: The statement "Every \( \Pi^1_1 \) subset of \( V \) has a perfect subset" is not a theorem of \( ZFC \).

But this gives us more: By \( \Pi^1_1 \)-absoluteness, the \( \Pi^1_1(a) \) set \( A \) constructed in 10.15 is \( \Pi^1_1(a) \) in the sense of \( V \).

Now assume \( \omega^V_1 \) is a cardinal successor in \( L \). So in \( L \) there is a cardinal \( \kappa \) such that \( \omega^L_1 = \kappa^L \). But \( \kappa \) is countable in the sense of \( V \), so working in \( V \) there is some \( \alpha \in W \) such that \( \alpha \) codes the order-type \( \kappa \). This means:

\[
\omega^L_1(\alpha) = \omega^V
\]

Then the set \( A \) satisfies the assumption of 10.11 in the sense of \( V \), so \( A \) is a \( \Pi^1_1(a) \) set which is a counterexample to PSP in the sense of \( V \). So we have proved:

10.17. Theorem: If every \( \Pi^1_1 \)-set has the PSP then \( \omega^V_1 \) is a limit cardinal in \( L \). (More generally, in \( L[\kappa] \) for every \( \kappa \in W \).)

10.18. Corollary: Assume \( \omega^V_1 \) is regular. Then

\[
PSP(\Pi^1_1) \implies Con(\neg \exists \kappa \text{ inaccessible})
\]

That is, for instance it is not possible to prove

\[
Con(ZFC) \implies Con(ZFC + PSP(\Pi^1_1) + \omega^V_1 \text{ is regular})
\]

10.19. Remark: It is possible to prove

\[
Con(ZFC) \implies Con(ZFC + PSP(\Pi^1_1) + \omega^V_1 \text{ is singular})
\]

(Theorem of J. Truss)

More on Fine Structure

10.20. Theorem: \( (\mathcal{L}_\infty, \in) \), \( \mathcal{L}_\infty \)-uniformization \( \setminus \) structures (\( \mathcal{L}_\infty \), \( \in \)) admit uniform \( \mathcal{L}_\infty \)-uniformization, i.e., if \( \phi(u, v) \) is a \( \mathcal{L}_\infty \)-formula in \( \mathcal{L}_\infty \) such that \( \phi(u, v) \) holds then \( \exists z \in V \) \( \phi(z, t) \) holds for every \( \in \in V \).
\( \phi(v) \) defines a function which uni
formizes the relation

\[ R_v = \{ (x, y) \mid M \models \phi_v(x, y) \} \text{ when } M = (S, A, B). \]

The \( \tilde{\phi} \) is obtained from \( \phi \) in a recursive way.

\[ \tilde{\phi}(v, w) = \sum_{w} \phi_0(v, u, w) \text{ when } \phi_0 \triangleq \Delta_0. \]

We then let

\[ \tilde{\phi}(v, w) \equiv (\exists v') (v', w') \text{ s.t. } \phi(v, v', w') \]

\( \tilde{\phi} \) defines a function (partial) \( f(v) \) with output \( v, w' \).

If \( \eta(v) \) is the \( \leq \) uniformly function for \( \tilde{\phi}(v, w) \) then \( \eta(v) \) is the uniformizing function for \( \tilde{\phi} \) and it is easy to verify that its definition only depends on \( \phi_0 \).

\[ \tilde{\phi}(v, w) \equiv \left( \exists v' \exists v'' (v, w) \right) \]

\[ \left\{ \begin{array}{l}
\text{Deny} \times S \subseteq S^A \times W = \tilde{W}^A \\
\langle v, w \rangle \in S \\n\phi_0(v, u, w) \\
A_0 \\
(\exists v', w' \in S) \left( \langle v, w \rangle \in W \rightarrow \tilde{\phi}_0(v, u, w', w) \right) \\
\langle v', w' \rangle \leq v, w \end{array} \right. \]

This proves \( \tilde{\phi} \in \Sigma_1. \)

\[ \square \quad T 10.20 \]

10.24 Remark The task for the Fine Structure Theorem is to do this for complexities higher than \( \Sigma_1 \). Point 10.24 would be false for \( \Sigma_n \) when \( n > 1 \). Being hierarchically does not work here. Denote an \( \Delta_n \) as shown by Jensen. One way is to replace \( \Delta_n \) with a different one. Jensen: \( \Sigma_n \)-hierarchy, \( \Delta_n \)-hierarchy. One way is to replace with some different one. Jensen: \( \Sigma_n \)-hierarchy, \( \Delta_n \)-hierarchy.

Before going further: Switch to finitely with a single free variable \( v \). Let \( (\phi_i(v) \mid i \in \omega) \) be a fixed recursive enumeration of such finitely. We let

\[ \phi^*_i(v) \equiv \phi_i \left( \left[ \phi^*_j \right] v \rightarrow x \right) \]

We can go back to our arguments by considering \( \langle v, x \ldots x \rangle \) as \( v \).

Using 10.20, we can \( \Sigma_1 \)-uniformize the relation with definition

\[ \phi^*_i(v) \equiv \phi^*_i (v, u) \text{ (in a argument, } v \text{ is the output) } \]

This will give us a partial function \( h_v(c, u) \) such that for every structure \( M = (S, A, B) \):
10.22. Corollary. Assume $M = \langle \mathcal{F}^a, B \rangle$ and $X \subseteq M$. Let
\[ Y = \{ h^a_m(i, s) \mid s \in B \} \] be $h^a_m(X)$
Then $Y \subseteq \mathcal{F}^a$. (Max period: $\langle Y, E, B, Y \rangle \in \mathcal{F}^a$)
Proof. Exercise, like the proof of Prop. 10.3. But here we can see the properties of $h^a_m$.
\[ \square \]

10.23. Definition. Let $M = \langle \mathcal{F}^a, B \rangle$ be acceptable. (This means: $\mathcal{F}^a$
\[ \subseteq \mathcal{F}^a \]) and acceptable, as $B$ does not contribute to the hierarchy $\mathcal{F}^a$. The first projective
$\mathcal{F}^a$ of $M$ is the smallest ordinal $\alpha$ s.t. there exists a $\mathcal{F}^a(M)$
set $A$ such that $M \subseteq \mathcal{F}^a(M)$ and $A \subseteq \mathcal{F}^a(M)$.
(Note: $\mathcal{F}^a \subseteq \mathcal{F}^a$)

10.24. Definition. Let $M$ be as in Prop. 10.23. The set $P^a_M$ is defined as
follows:
\[ p \in P^a_M \iff p \in \{ \text{On} \cap M \} \] and $p$ is a $\mathcal{F}^a(M)$-$A$ such that
$A \subseteq \mathcal{F}^a(M)$ and $A \cup P^a_M \subseteq M$.

The elements of $P^a_M$ are called good parameters.

10.25. Proposition. $h^a_m(\text{On} \cap M) = M$. So if $A \subseteq \mathcal{F}^a(M)$
in some parameter $q \in M$ then $A \subseteq \mathcal{F}^a(M)$ in some $p \in P^a_M$;
it suffices to take $p$ so that $l = h^a_m(i, p)$.
Proof. Exercise.

10.26 Definition. Again $M$ is as in Prop. 10.23. The set $R^a_M$ is defined as
follows:
\[ p \in R^a_M \iff p \in \{ \text{On} \cap M \} \] and $h^a_m(P^a_M \cup P^a_M) = M$.

10.27. Proposition. $M$ as above
(a) $P^a_M \neq \emptyset$
(b) $R^a_M \subseteq P^a_M$
Proof. (a) Trivial. (b) This is done by the diagonalisation argument from §10.8. One just needs to check that if in that argument we take \( \mathcal{E}_2 \) then the diagonal set \( \bar{\delta} \) is also \( \mathcal{E}_3 \). \( \Box \) §10.29.

10.29 Remark. Recall we have the set-like well-ordering on \([0^\omega]^\omega\) defined by:

\[
p <^* q \iff \exists \alpha \in \mathbb{N}_+ . \exists \mathbf{p}, \mathbf{q} : \mathbf{p} <^* \mathbf{q} \text{ s.t. } p \cdot (\alpha + 1) = q \cdot (\alpha + 1)
\]

10.30 Def. The first standard parameter of \( M \) is defined as follows:

\[
\mathbf{p}_0 = \text{the } <^* - \text{least element of } \mathbf{p}_0
\]