MATH 281C SPRING 2020 L5

10.31. Remark Recall: for $M = (J^A_\alpha, B)$ acceptable, amenable and $p \in [\omega\alpha]^{<\omega}$:

$p \in P^1_M \iff$ There is a set X which is $\Sigma_1(M)$ in p s.t. $X \cap \rho^1_M \notin M$

$\rho^1_M =$ the least ordinal ρ s.t. there is a set X which is $\underset{\sim}{\Sigma}_1(M)$ s.t. $X \cap \rho \notin M$.

$p^1_M =$ the $<^*$-least element of P^1_M

$p \in R^1_M \iff h^1_M(\rho^1_M \cup \{p\}) = M$

10.32 Proposition $M = (J^A_\alpha, B)$ as before (i.e. is acceptable)

(a) ρ^1_M is a Σ_1-cardinal with respect to M. This means there is no $\underset{\sim}{\Sigma}_1(M)$ surjection $f : \rho \to \rho^1_M$ for any $\rho < \rho^1_M$

(b) If $\tau \in M$ is regular in the sense of M then $J^A_\tau \models ZFC^-$

10.33. Definition Again $M = \langle J^A_\alpha, B\rangle$. We say that M is **1-sound** iff $R^1_M = P^1_M$.

Recall: $R^1_M \subseteq P^1_M$ (P 10.27 (b)).

10.34. Proposition Let $M = \langle J^A_\alpha, B\rangle$. Then M is 1-sound iff $p^1_M \in R^1_M$.

Proof $\Rightarrow$ trivial

$\Leftarrow$ For a contradiction, assume $R^1_M \subsetneqq P^1_M$ and $p^1_M \in R^1_M$.

Let $p =$ the $<^*$-least element of $P^1_M \setminus R^1_M$

p exists as $R^1_M \subsetneqq P^1_M$ and $p \neq p^1_M$ because we are assuming $p^1_M \in R^1_M$. Also $p^1_M <^* p$ by definition. Since $p^1_M \in R^1_M$, we have some $\xi < \rho^1_M$ s.t. $p = h^1_M(i, \langle \xi, p^1_M\rangle)$ some $i \in \omega$. Let

$X = h^1_M(\rho^1_M \cup \{p\})$

Then $X \prec_{\Sigma_1} M$ but $p^1_M \notin X$ otherwise $p \in R^1_M$, which is not.

Now

(1) $M \models \underbrace{(\exists v)(v \in [On]^{<\omega} \wedge v <^* p \wedge \text{“} p = h^1_M(i, \langle \xi, v\rangle) \text{”})}_{\Sigma_1}$

So there is a $v \in X$ as in the RHS of (1). Denote this v by p_1. Now notice (1) holds with p_1 in place of p but with some possibly different $i \in \omega$ and $\xi < \rho^1_M$. This way we get some $p_2 \in X$ satisfying (1). Etc. If we iterate this procedure ω times, we get an infinite sequence

$$p >^* p_1 >^* p_2 >^* \dots >^* p_i >^* \dots$$

Contradiction, as $<^*$ is a well-ordering. $\square$ P 10.34.

10.35. Proposition Each J_α is 1-sound.

Proof Let $X = h^1_{J_\alpha}(\rho^1_{J_\alpha} \cup \{p^1_{J_\alpha}\})$ and $\sigma : J_{\bar\alpha} \longrightarrow J_\alpha$ be the inverse of the collapsing map of X. Let $p = \sigma^{-1}(p^1_{J_\alpha})$. Notice σ is Σ_1-preserving and $\sigma \restriction \rho^1_{J_\alpha} = id$. Also $\bar\alpha \leq \alpha$ trivially. If Z is $\Sigma_1(J_\alpha)$ in $p^1_{J_\alpha}$ s.t. $Z \cap \rho^1_{J_\alpha} \notin J_\alpha$ then Z is $\Sigma_1(J_{\bar\alpha})$ in p by the same definition. Hence $\bar\alpha = \alpha$. Then $J_\alpha = h^1_{J_\alpha}(\rho^1_{J_\alpha} \cup \{p\})$

So $p \in R^1_{J_\alpha} \Rightarrow p \in P^1_{J_\alpha}$. But $p \leq^* p^1_{J_\alpha}$ as $\sigma(p) = p^1_{J_\alpha}$.

Hence $p = p^1_{J_\alpha}$. Since $p \in R^1_{J_\alpha}$ then J_α is 1-sound by P 10.34.

$\square$ P 10.35.

CONSTRUCTIONS OF $\square$ SEQUENCES IN L

From now on work inside L. Let κ be a cardinal. Our intention is to construct a $\square(\kappa^+)$-sequence and a $\square_\kappa$-sequence. Let $\tau \in (\kappa, \kappa^+)$ be s.t. $J_\tau \prec J_{\kappa^+}$. We define

$\beta(\tau)$ = the least β s.t. $J_{\beta+1} \models \tau$ is not a cardinal.

We say that $J_{\beta(\tau)}$ is the collapsing level of L for τ. There is surjection $f : \kappa \longrightarrow \tau$ which is $\underset{\sim}{\Sigma}_n(J_{\beta(\tau)})$ for some $n \in \omega$. Define

$n(\tau)$ = the least $n \in \omega$ s.t. there is a $\underset{\sim}{\Sigma}_n(J_{\beta(\tau)})$ definable surjection $f : \kappa \longrightarrow \tau$.

Caution If $n > 1$ we need to use Jensen's or Steel's hierarchy mentioned last time. We will only deal with the case $n(\tau) = 1$

Notice: if $n(\tau) = 1$ then $\rho^1_{J_{\beta(\tau)}} = \kappa$, so $h^1_{J_{\beta(\tau)}}(\kappa \cup \{p^1_{J_{\beta(\tau)}}\}) = J_{\beta_\tau}$ by P 10.35.

Let

S = the set of all $\tau \in (\kappa, \kappa^+)$ s.t.

(i) $J_\tau \prec J_{\kappa^+}$ (Notice $\beta(\tau) > \tau$ in this case)

(ii) $n(\tau) = 1$

(1) S is stationary (Exercise)

(2) Introduce the following notation for $\tau \in S$

- $h_\tau = h^1_{J_{\beta(\tau)}}$
- $p_\tau = p^1_{J_{\beta(\tau)}}$

For $\tau \in S$ define a set B_τ as follows:

$\bar\tau \in B_\tau$ iff
(a) $\bar\tau \in S$
(b) There is a Σ_0-preserving map
$$\sigma : J_{\beta(\bar\tau)} \longrightarrow J_{\beta(\tau)}$$
such that
(i) $\sigma \upharpoonright \bar\tau = id \upharpoonright \bar\tau$
(ii) $\sigma(\bar\tau) = \tau$
(iii) $\sigma(p_{\bar\tau}) = p_\tau$

10.36 Theorem The family $(B_\tau \mid \tau \in S)$ has the following properties

(a) B_τ is a closed subset of τ, and if $cf(\tau) > \omega$ then B_τ is unbounded
(b) If $\bar\tau \in B_\tau$ then $B_{\bar\tau} = B_\tau \cap \bar\tau$
(c) The family $(B_\tau \mid \tau \in S)$ is not threadable

10.37. Theorem The family $(B_\tau \mid \tau \in S)$ can be turned into a $\square(\kappa^+)$-sequence on S by purely combinatorial manipulations

Proof (Exercise)

We now fix more notation and background. Recall that the statement
$$y = h^1_M(i, \langle x, q \rangle)$$
is uniformly Σ_1 for acceptable amenable J_i structures M, so working in L, we can fix a Σ_0-formula $\phi_0(u_0, u_1, u_2, u_3, v)$ such that for every $\beta \in On$, $i \in \omega$ and $x, q, y \in J_\beta$ we have

$$y = \hat{h}_{J_\beta}(i, \langle x, q\rangle) \quad \text{iff} \quad J_\beta \models (\exists v)\,\phi_0(i, x, q, y, v)$$

<u>10.38. Proposition</u> The map $\sigma : J_{\beta(\bar\tau)} \longrightarrow J_{\beta(\tau)}$ from (6) in the definition of B_τ is <u>unique</u>. Therefore we will denote it by $\sigma_{\bar\tau\tau}$, or simply by $\sigma_{\bar\tau}$ if τ is clear from the context.

<u>Proof</u> Let $x \in J_{\beta(\bar\tau)}$. By soundness

$$(1) \qquad x = h_{\bar\tau}(i, \langle \xi, p_{\bar\tau}\rangle) \quad \text{for some } i \in \omega,\ \xi < \kappa.$$

This statement is Σ_1, and recall that Σ_1-statements are preserved upward by Σ_0-maps. So we have

$$(2) \qquad \sigma_{\bar\tau}(x) = h_\tau(i, \langle \xi, p_\tau\rangle)$$

as i, ξ are not moved by $\sigma_{\bar\tau}$, and $p_{\bar\tau}$ is mapped to p_τ.
More detailled explanation: We can do this as h_M have uniform Σ_1-definition. So we are actually applying $\sigma_{\bar\tau}$ to the formula $(\exists v)\,\phi_0(i, \xi, p_{\bar\tau}, x, v)$ and get

$$J_{\beta(\tau)} \models (\exists v)\,\phi_0(i, \xi, p_\tau, \sigma_{\bar\tau}(x), v)$$

The point: the formula $(\exists v)\,\phi_0(u_0, u_1, u_2, u_3, v)$ defines a function with value u_3. Hence it does not matter what v is: if $J_{\beta(\tau)} \models \phi_0(i, \xi, p_\tau, y, z)$

and $J_{\beta(\tau)} \models \phi_0(i, \xi, p_\tau, y', z')$

Then $y = y'$ and this value is then $\sigma_\tau(x)$. □ P10.38

<u>10.39. Proposition</u> $\sigma_{\bar\tau}$ is <u>not</u> Σ_1-preserving, hence <u>not</u> cofinal.

<u>Proof</u> Since $J_{\beta(\tau)}$ is sound, we have

$$(1) \qquad \bar\tau = h_{J_{\beta(\tau)}}(i, \langle \xi, p_\tau\rangle) \quad \text{some } i \in \omega,\ \xi < \kappa.$$

So $i, \xi, p_\tau \in \operatorname{rng}(\sigma_{\bar\tau})$. The statement

$$(2) \qquad (\exists v)\ v = h_{J_{\beta(\tau)}}(i, \langle \xi, p_\tau\rangle)$$

is Σ_1. So if $\sigma_{\bar\tau}$ were Σ_1-preserving then there would be v as in (2) in the range of $\sigma_{\bar\tau}$. But that v

can only be equal to $\bar{\tau}$, and $\bar{\tau} \notin \mathrm{rg}(\sigma_{\bar{\tau}})$ by (b) clauses (i), (ii) in the definition of $B_{\bar{\tau}}$. Contradiction.

$\square$ P 10.39.

<u>10.40 Remark</u> So if we take $\tau, \bar{\tau}, i, \zeta$ as in the proof of P 10.39 then we have an example where

$h_{\tau}(i, \langle \zeta, p_{\tau} \rangle)$ is defined and $= \bar{\tau}$ but

$h_{\bar{\tau}}(i, \langle \zeta, p_{\tau} \rangle)$ is undefined