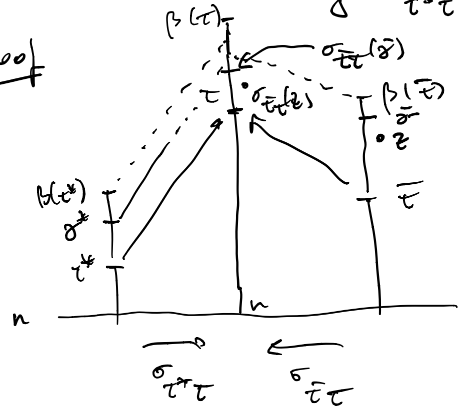
M281C SPRING 2020 L6

10.41. Prop Assume $\tau^* < \bar\tau$ and $\tau^*, \bar\tau \in B_\tau$ then
$$rng(\sigma_{\tau^*\tau}) \subseteq rng(\sigma_{\bar\tau\tau})$$

Proof



We first prove:

(1) $\sup \sigma_{\tau^*\tau}[\omega\beta(\tau^*)] < \sup \sigma_{\bar\tau\tau}[\omega\beta(\bar\tau)]$

Proof Suppose not, so we have $\geq$ as on the picture.
We know: There are $i \in \omega$, $\zeta < n$ s.t. $\tau^* = h_{\bar\tau}(i, \langle \zeta, p_{\bar\tau} \rangle)$

Recall we have formula $\varphi_0(u_0, u_1, u_2, u_3, v)$ which is Δ_0 and

(2) $J_{\beta(\bar\tau)} \models \varphi_0(i, \zeta, p_{\bar\tau}, \tau^*, z)$ for some $z \in J_{\beta(\bar\tau)}$

Apply $\sigma_{\bar\tau\tau}$ to (2) and get

(3) $J_{\beta(\tau)} \models \varphi_0(i, \zeta, p_\tau, \tau^*, \sigma_{\bar\tau\tau}(z))$

Here i, ζ, τ^* are not moved as they are $< \bar\tau$. Now $z \in S_{\bar\gamma}$ for some $\bar\gamma < \omega\beta(\bar\tau)$. By our assumption, we can find $\gamma < \omega\beta(\tau^*)$ s.t. $\sigma_{\tau^*\tau}(\gamma) \geq \sigma_{\bar\tau\tau}(\bar\gamma)$. Then from (3) we get

(4) $J_{\beta(\tau)} \models (\exists z' \in \sigma_{\tau^*\tau}(S_\gamma))\, \varphi_0(i, \zeta, p_\tau, \tau^*, z')$

(5) $J_{\beta(\tau)} \models \underbrace{(\exists z' \in \sigma_{\tau^*\tau}(S_\gamma))\, \exists \eta \in \sigma_{\tau^*\tau}(S_\gamma)\, \varphi_0(i, \zeta, p_\tau, \eta, z')}_{\Sigma_0}$

Can be pulled back by $\sigma_{\tau^*\tau}$. So $rng(\sigma_{\tau^*\tau})$ has an element like η in (5). But there is only one η that can make (5) true, namely $\eta = \tau^*$, by (2). Contradiction, as $\tau^* = cr(\sigma_{\tau^*\tau})$. $\square$ (1)

Now the proof of the Proposition is similar. If $x \in rng(\sigma_{\tau^*\tau})$ then we have $i \in \omega$ and $\zeta < n$ s.t.
$$\bar x = h_{\tau^*}(i, \langle \zeta, p_{\tau^*} \rangle) \text{ where } \sigma_{\bar\tau\tau}(\bar x) = x.$$

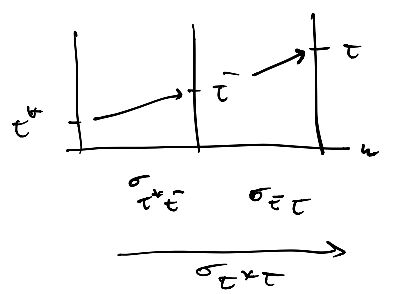
We need to see that $h_{\bar\tau}(i, \langle \xi, p_{\bar\tau} \rangle)$ is defined. If we have this then we get

$$\sigma_{\bar\tau\tau}(h_{\bar\tau}(i, \langle \xi, p_{\bar\tau}\rangle)) = h_\tau(i, \langle \xi, p_\tau\rangle) = \sigma_{\tau^*\tau}(h_{\tau^*}(i, \langle \xi, p_{\tau^*}\rangle)) = x$$

Here we proceed similarly as in the proof of (1), using (1) to turn the statement "$x = h_\tau(i, \langle \xi, p_\tau\rangle)$" into a Δ_0-statement. (Exercise) $\square$P10.41.

10.42. Def If $\tau^* < \bar\tau$ and $\tau^* \le \bar\tau$ we define $\sigma_{\tau^*\bar\tau}$ by

$$\sigma_{\tau^*\bar\tau} = \sigma_{\bar\tau\tau}^{-1} \circ \sigma_{\tau^*\tau}$$



10.43 Prop The maps $\sigma_{\tau^*\bar\tau}$ from Def 10.42 have the following properties.

(i) $\sigma_{\tau^*\bar\tau}$ is Σ_0-preserving

(ii) $\sigma_{\tau^*\bar\tau} \restriction \tau^* = \mathrm{id} \restriction \tau^*$

(iii) $\sigma_{\tau^*\bar\tau}(\tau^*) = \bar\tau$

(iv) $\sigma_{\tau^*\bar\tau}(p_{\tau^*}) = p_{\bar\tau}$

Proof of T 10.36 (b) If $\bar\tau \in B_\tau$ then $B_{\bar\tau} = B_\tau \cap \bar\tau$

$\subseteq$ If $\tau^* \in B_{\bar\tau}$ then it is easy to check, using P 10.43 that $\sigma_{\bar\tau\tau} \circ \sigma_{\tau^*\bar\tau}$ witnesses that $\tau^* \in B_\tau$, as it has the properties in (b) of the definition of B_τ. So in fact we have

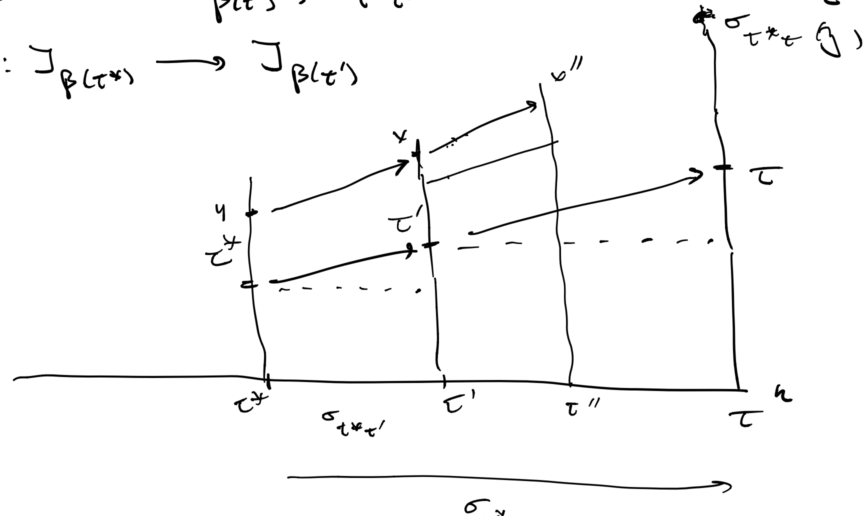
$$\sigma_{\bar\tau\tau} \circ \sigma_{\tau^*\bar\tau} = \sigma_{\tau^*\tau}$$

$\supseteq$ Now assume $\tau^* \in B_\tau$ and $\tau^* < \bar\tau$. Then by P 10.41 $\mathrm{rng}(\sigma_{\tau^*\tau}) \subseteq \mathrm{rng}(\sigma_{\bar\tau\tau})$ and $\sigma_{\tau^*\bar\tau} = \sigma_{\bar\tau\tau}^{-1} \circ \sigma_{\tau^*\tau}$ exists. It has the properties as in (b) in the def of $B_{\bar\tau}$ ($\bar\tau$ plays the role of τ in that def). $\square$ T 10.36 (b).

Proof of T 20.36 (b) the part about being closed. B_τ is a closed subset of τ.

Proof Assume $\bar\tau < \tau$ is a limit point of B_τ. We prove $\bar\tau \in B_\tau$. From the above we have the following commuting diagram: $\langle J_{\beta(\tau')}, \sigma_{\tau^*\tau'} \mid \tau^* \leq \tau' \ \&\ \tau^*, \tau' \in B_\tau \cap \bar\tau \rangle$

$\sigma_{\tau^*\tau'} : J_{\beta(\tau^*)} \longrightarrow J_{\beta(\tau')}$



$\mathbb{D} = (D, E)$

We take the direct limit of this diagram. This is defined as follows: Let

$D_0 = \{\langle \tau', x\rangle \mid x \in J_{\beta(\tau')}\}$

$I = \{\langle\langle \tau', x\rangle, \langle \tau^*, y\rangle\rangle \in D_0 \times D_0 \mid (\tau^* \leq \tau' \wedge \sigma_{\tau^*\tau'}(y) = x)$
$\vee (\tau' \leq \tau^* \wedge \sigma_{\tau'\tau^*}(x) = y)\}$

Then

(1) I is an equivalence relation on D_0 (Exercise)

and we let

(2) $D = D_0 / I$

then we define a binary relation E on D by

(3) $\langle \tau', x\rangle\ E\ \langle \tau^*, y\rangle$ iff

$(\tau^* \leq \tau' \wedge \sigma_{\tau^*\tau'}(x) \in y) \vee (\tau' \leq \tau^* \wedge \sigma_{\tau'\tau^*}(y) \in x$

(4) E is well-founded.

<u>Proof of (4)</u>: Define a map $\sigma : \mathbb{D} \to J_{\beta(\tau)}$ by

$$\sigma([\tau^*, x]) = \sigma_{\tau^*\tau}(x)$$

and notice: this definition does not depend on the choice of representative.

The map σ is structure-preserving:

$$[\tau^*, x] \mathrel{E} [\tau', y] \Rightarrow \sigma([\tau^*, x]) \in \sigma([\tau', y])$$

Since $J_{\beta(\tau)}$ is transitive, it follows that E is well-founded. $\square$ (4)

(5) E is extensional (Exercise)

So we can transitively collapse $\mathbb{D}$ to get a transitive structure M. For each $\tau^* \in B_\tau \cap \tau$ we have the canonical map

$$\tilde{\sigma}_{\tau^*} : (J_{\beta(\tau^*)}, \in) \to \mathbb{D}$$

defined by

$$\tilde{\sigma}_{\tau^*}(x) = [\tau^*, x]$$

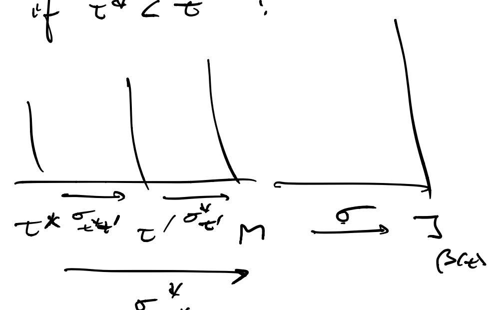
(6) $\tilde{\sigma}_{\tau^*}(x)$ is Σ_0-preserving (Exercise)

If $\pi : \mathbb{D} \to (M, \in)$ is the collapsing isomorphism then $\pi \circ \tilde{\sigma}_{\tau^*} : (J_{\beta(\tau^*)}, \in) \to (M, \in)$ is Σ_0-preserving.

Denote $\sigma^*_{\tau^*} = \pi \circ \tilde{\sigma}_{\tau^*}$. Then if $\tau^* < \tau'$:

(7) $\sigma^*_{\tau^*} = \sigma^*_{\tau'} \circ \sigma_{\tau^*\tau'}$

(8) $\sigma^*_{\tau^*}$ is Σ_0-preserving.



<u>Next</u>:

(9) M is of the form J_β

This follows from the existence of the map σ and the fact that the statement "I am a J_β" is a Q-statement.

We also have:

(10) $$\bigcup_{\tau^* \in B_\tau \cap \bar{\tau}} \operatorname{rng}(\sigma^*_{\tau^*}) = M$$

The fact that the statement "I am a J_β" is true for M also follows from the following

(11) Assume we have a diagram of structures $\langle J_{\alpha_i}, \pi_{ij} \mid i \leq j < \delta \rangle$ with $\pi_{ij} : J_{\alpha_i} \longrightarrow J_{\alpha_j}$ all Σ_0-preserving. Also assume M is transitive and we have Σ_0-preserving maps $\pi_i : J_{\alpha_i} \to M$ s.t.

(i) the maps π_{ij} commute

(ii) if $i<j$ then $\pi_i = \pi_j \circ \pi_{ij}$.

(iii) $\bigcup_{i<\delta} \operatorname{rng}(\pi_i) = M$

Then if φ is a Q sentence and $J_{\alpha_i} \models \varphi$ for a tail-end of $i<\delta$ then $M \models \varphi$.