M281C SPRING 2020 L7

Recall: We are proving B_τ is closed. We picked $\bar{\tau}<\tau$ a limit pt of B_τ and WTS $\bar{\tau}\in B_\tau$. We had the commutative diagram $(J_{\beta(\tau^*)}, \sigma_{\tau^*\tau'} \mid \tau^*<\tau' \text{ in } B_\tau\cap\bar{\tau})$. We took the direct limit of this diagram and proved it is well-founded and the transitive collapse is of the form J_β for some β. We obtain the following commutative diagram

$$\begin{tikzcd}
 & & & J_{\beta(\tau)} \\
J_{\beta(\tau^*)} \arrow[r, "\sigma_{\tau^*\tau'}"'] \arrow[urrr, bend left, "\sigma_{\tau^*\tau}"] \arrow[rrr, bend right=40, "\sigma^*_{\tau^*}"'] & J_{\beta(\tau')} \arrow[r] \arrow[urr, "\sigma_{\tau'\tau}"] \arrow[rr, bend right, "\sigma^*_{\tau'}"'] & \cdots \arrow[r] & J_\beta \arrow[u, "\sigma^*"']
\end{tikzcd}$$

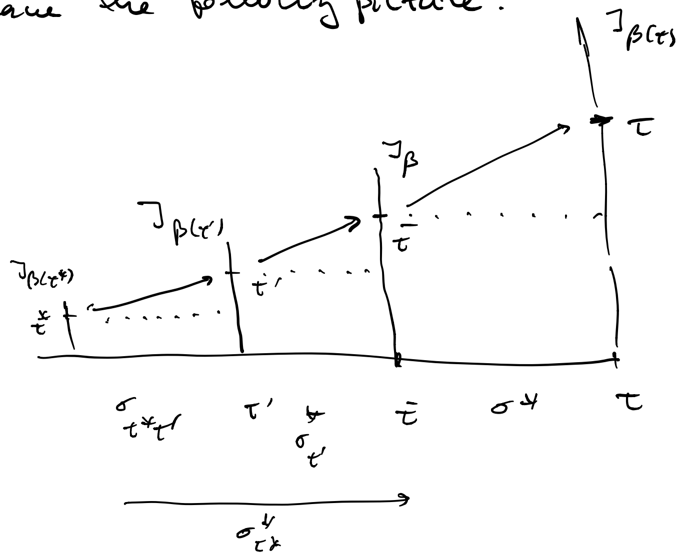
σ^* ... corresponds to $\sigma: D \to J_\tau$ after collapsing

All maps are Σ_0-preserving. We also get

(12) $\sigma^*_{\tau^*}(\tau^*)=\bar{\tau}$ all $\tau^*\in B_\tau\cap\bar{\tau}$ and $\sigma^*(\bar{\tau})=\tau$

Proof HW.

So we have the following picture:



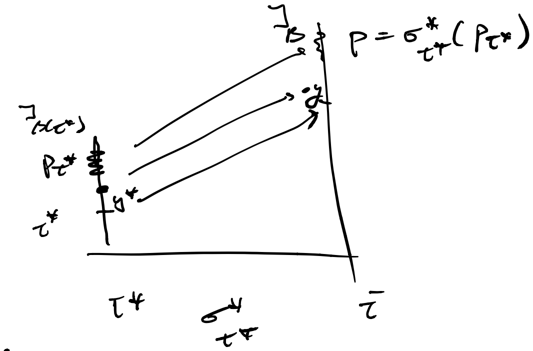
(13) $\bar\tau = \kappa^{+ J_\beta}$

Proof This is because $\bar\tau$ is a cardinal in J_β and no ordinal in the in the interval $(\kappa, \bar\tau)$ is a cardinal in J_β.

To see the former: $\sigma^*_{\tau^*}(\tau^*) = \bar\tau$ for all $\tau^* \in B_\tau \cap \bar\tau$ and this is a cofinal set of ordinals in $\bar\tau$; now apply (11).

To see the latter: Notice no ordinal in the interval (κ, τ^*) is a cardinal in $J_{\beta(\tau^*)}$ for any $\tau^* \in B_\tau \cap \bar\tau$, and $J_{\bar\tau} = \bigcup\{J_{\beta(\tau^*)} \mid \tau^* \in B_\tau \cap \bar\tau\}$. $\square$ (13)

(14) $\rho^1_{J_\beta} = \kappa$



Let $p = \sigma^*_{\tau^*}(p_{\tau^*})$ and notice this value does not depend on the choice of $\tau^* \in B_\tau \cap \bar\tau$, by the commutativity of the diagram.

Recall $\phi_0(u_0, u_1, u_2, u_3, v)$ is our Δ_0-formula which defines the canonical Σ_1-Skolem function, i.e.

$y = h_\alpha(i, \langle x, p\rangle)$ iff $J_\alpha \models \phi_0(i, x, p, y, z)$ for some z

Now pick $y \in J_\beta$. By (10) there is some $\tau^* \in B_\tau \cap \bar\tau$ and some $y^* \in J_{\beta(\tau^*)}$ s.t. $y = \sigma^*_{\tau^*}(y^*)$. By soundness there an $i \in \omega$ and $\zeta \in \kappa$ s.t.

$y^* = h_{\tau^*}(i, (\zeta, p_{\tau^*}))$, i.e. $J_{\beta(\tau^*)} \models \phi_0(i, \zeta, p_{\tau^*}, y^*, z)$ for some z

Now apply $\sigma^*_{\tau^*}$:

$$J_\beta \models \phi_0(i, \zeta, \underbrace{p}_{\sigma^*_{\tau^*}(p_{\tau^*})}, \overbrace{y}^{\sigma^*_{\tau^*}(y^*)}, \sigma^*_{\tau^*}(z))$$

Hence

$$h_{J_\beta}(i, \langle \zeta, p \rangle) = y$$

This tells us that $h_{J_\beta}(\kappa \cup \{p\}) = J_\beta$. It follows that $\rho^1_{J_\beta} = \kappa$ which proves (14). $\square$ (14)

Moreover, the proof of (14) gives:

(15) $p \in R^1_{J_\beta}$

Now we know

(16) $\beta = \beta(\bar\tau)$, so $\bar\tau \in S$

Regarding $\bar\tau \in S$: We just proved that $\rho^1_{J_{\beta(\bar\tau)}} = \kappa$; to see $J_{\bar\tau} \prec J_{\kappa^+}$: Exercise.

To see that $\bar\tau \in B_\tau$ it suffices to show:

(17) $\sigma^* = \sigma_{\bar\tau\tau}$.

Recall: σ^* is defined by

$$\sigma^*(\sigma^*_{\tau^*}(x)) = \sigma_{\tau^*\tau}(x)$$

$$\begin{tikzcd} & J_{\beta(\tau)} \\ J_{\beta(\tau^*)} \arrow[ur, "\sigma_{\tau^*\tau}"] \arrow[r, "\sigma^*_{\tau^*}"'] & J_{\beta(\bar\tau)} \arrow[u, "\sigma^*"'] \end{tikzcd}$$

and this value does not depend on the choice of $\tau^* \in B_\tau \cap \bar\tau$.

So: If $\zeta < \bar\tau$, pick $\tau^* \in B_\tau \cap \bar\tau$ s.t. $\tau^* > \zeta$. Then

$$\sigma^*(\zeta) = \sigma^*(\sigma^*_{\tau^*}(\zeta)) = \sigma_{\tau^*\tau}(\zeta) = \zeta$$

$$\uparrow$$

$$\sigma^*_{\tau^*} \restriction \tau^* = id \restriction \tau^*$$

This shows: $\sigma^* \restriction \bar\tau = id$

Similarly we show: $\sigma^*(\bar\tau) = \tau$ and $\sigma^*(p) = p_\tau$

To complete the proof, it suffices to show:

(18) $p = p_{\bar\tau}$

Proof If not then, $p_{\bar{\tau}} <^* p$ since we know that $p \in P^1_{J_{\beta(\bar{\tau})}}$. By soundness then we can find $i \in \omega$ and $\zeta < n$ s.t.

$$(19) \qquad p = h_{\bar{\tau}}(i, \langle \zeta, p_{\bar{\tau}} \rangle)$$

Now again using (20) find $\tau^* \in B_\tau \cap \bar{\tau}$ large enough that $p_{\bar{\tau}} \in \mathrm{rng}(\sigma^*_{\tau^*})$, say $p_{\bar{\tau}} = \sigma^*_{\tau^*}(q_{\tau^*})$ for all sufficiently large $\tau^* \in B_\tau \cap \bar{\tau}$. The maps $\sigma^*_{\tau^*}$ are Σ_0-preserving, but similarly as in the proof of (14) we can find $\tau^* \in B_\tau \cap \bar{\tau}$ large enough so that we can pull the statement (19) back to $J_{\beta(\tau^*)}$, getting

$$p_{\tau^*} = h_{\tau^*}(i, \langle \zeta, q_{\tau^*} \rangle) \quad \text{and} \quad q_{\tau^*} <^* p_{\tau^*}$$

This would mean that $q_{\tau^*} \in R^1_{J_{\beta(\tau^*)}}$ (Exercise)

Since $q_{\tau^*} <^* p_{\tau^*}$, this is a contradiction. $\boxtimes$(18).

Thus completes the proof that $\bar{\tau} \in B_\tau$, and hence the proof that B_τ is closed.

As mentioned before, we do a proof of unboundedness of B_τ for $cf(\tau) > \omega$ later.

10.44 Remark Most of the properties of $\bar{\tau}$, J_β could be also proved using the map σ^* (Exercise.)
But doing it the above way enables us to establish these properties without relying on any extra information given by σ^*.

10.45. Theorem (V=L) The sequence $(B_\tau \,|\, \tau \in S)$ does not have a thread.

Proof Suppose there is a thread, call it C. Since S is stationary then $C \cap S$ is stationary in n^+ and we have the commutative diagram $(J_{\beta(\tau)}, \sigma_{\bar{\tau}\tau} \mid \bar{\tau} < \tau$ in $C \cap S)$.

Let $\mathbb{D}$ be the direct limit of this diagram.

(1) $\mathbb{D}$ is well-founded.

Proof Assume $\mathbb{D}$ is ill-founded. Then we can get a witness to ill-foundedness of the form $\{[\tau_n, \alpha_n] \mid n \in \omega\}$. Here $\alpha_n \in On \cap J_{\beta(\tau_n)}$ and $[\tau_{n+1}, \alpha_{n+1}] E [\tau_n, \alpha_n]$ all n. This translates as $\sigma_{\tau_n \tau_{n+1}}(\alpha_n) > \alpha_{n+1}$ all n. Because $cf(\kappa^+) > \omega$

we can find $\tau \in S \cap C$ s.t. $\tau > \tau_n$ all n. The above inequalities then can be turned, using the commutativity of the diagram into:

$$\sigma_{\tau_0 \tau}(\alpha_0) > \sigma_{\tau_1 \tau}(\alpha_1) > \sigma_{\tau_2 \tau}(\alpha_2) > \dots > \sigma_{\tau_n \tau}(\alpha_n) > \dots$$

Contradiction. $\square$ (1)

So the direct limit can be transitivized and we can as above prove:

- The transitive collapse of $\mathbb{D}$ is of the form J_β
- $\kappa^{+ J_\beta} = \kappa^+$
- $h_{J_\beta}(\kappa \cup \{p\}) = J_\beta$ when $p = \sigma_\tau(p_\tau)$

independent of $\tau \in C \cap S$

But this tells us κ^+ is collapsed to κ. Contradiction.

$\square$ T10.45

10.46. Theorem Now assume $(B_\tau \mid \tau \in C)$ is constructed in L when $C = \{\tau \in (\kappa, \kappa^+) \mid J_\tau \prec J_{\kappa^+}\}$ in the analogous way.
If $(B_\tau \mid \tau \in C)$ has a thread in V then $cf^V(\kappa^{+L}) = \omega$.

Proof Assume C^* is a thread in V for $(B_\tau \mid \tau \in C)$. As in the proof of T10.45. We have a commutative

diagram $(J_{\beta(\tau)}, \sigma_{\tau\tau'} \mid \tau < \tau' \text{ in } C^*)$. This diagram is typically not in L, but since $cf^V(\kappa^{+L}) > \omega$, the argument from the proof of T10.45 can be used to prove that its direct limit $\mathbb{D}$ is well-founded. So $\mathbb{D}$ can be transitivised and again as in the proof of T10.45 we show it is of the form J_β, $\kappa^{+L} = \kappa^{+J_\beta}$ and $h_{J_\beta}(\kappa \cup \{p_\beta\}) = J_\beta$.
So even though the diagram is not in L, its direct limit is in L. So we again get a collapsing function for κ^{+L} in L, a contradiction. $\square$T10.46.

<u>The last step</u>: How to turn the $\square(\kappa^+)$-sequence $(B_\tau \mid \tau \in C)$ into a $\square_\kappa$-sequence $(C_\tau \mid \tau \in C)$. Here $C = \{\tau \in (\kappa, \kappa^+) \mid J_\tau \prec J_{\kappa^+}\}$

<u>Fix $\tau \in C$</u> For $\zeta < \kappa$ let $X^\tau_\zeta = h_\tau(\{\zeta\} \cup \{p_\tau\})$

Notice X^τ_ζ is countable.

Now construct a sequence $(\tau_i \mid i < \gamma)$, $(\zeta_i \mid i < \gamma)$ as follows:

$\tau_0 = \min(B_\tau)$

If we have τ_i define ζ_i by
ζ_i = the least $\zeta < \kappa$ s.t. $X^\tau_\zeta \notin \operatorname{rng}(\sigma_{\tau_i \tau})$

If we have ζ_i define τ_{i+1} by
τ_{i+1} = the least $\tau' \in B_\tau$ s.t. $X^\tau_{\zeta_i} \subseteq \operatorname{rng}(\sigma_{\tau', \tau})$

If i is a limit then
$\tau_i = \sup_{i' < \tau} \tau_{i'}$ $\quad$ ($\tau_i \in B_\tau$ as B_τ is closed)

γ = the least i s.t. $\{\tau_{i'} \mid i' < i\}$ is cofinal in τ.

One inductively proves:

- Both sequences $(\tau_i \mid i < \gamma)$, $(\zeta_i \mid i < \gamma)$ are strictly increasing. So since $\zeta_i < \kappa$, we get $\gamma \leq \kappa$.
So letting
$$C_\tau = \{\tau_i \mid i < \gamma\}$$

we get $otp(C_\tau) \leq \kappa$.

Notice: If $cf(\tau) > \kappa$ then we know B_τ is unbounded in τ and so C_τ is unbounded in τ because all X_ξ^τ are countable.

Using Coherence of $(B_\tau)_\tau$ one shows that $(C_\tau)_\tau$ is also a coherent sequence, so this way $(C_\tau)_\tau$ will be a $\square_\kappa$-sequence.