M281C SPRING 2020 L8

11 BASICS OF FORCING

11.1. Background settings We will work with a model M of a sufficiently large fragment of ZFC. We will often have the full ZF or ZFC. Typically, M will be an universe, so we will often write V for M. M will be the model we will "force" over, i.e. we will have a poset $\mathbb{P} \in M$ and a filter G generic for $\mathbb{P}$ over M (in our previous terminology: G is $\mathcal{D}$-generic when $\mathcal{D} = \{D \in M \mid D \text{ is dense in } \mathbb{P}\}$) and we will "adjoin" G to M. In order that this can be done correctly we will make the following assumption: "Background assumption on generics."

(BAG) We have a model W of ZF such that

(1) M is a transitive structure in W

(2) For every $p \in \mathbb{P}$ there is a filter $G \in W$ generic for $\mathbb{P}$ over M such that $p \in G$

Point One can show that it is consistent to have this situation, assuming ZF is consistent. So we prove a metatheorem of the form

$$\mathrm{Con}(ZF) \Longrightarrow \mathrm{Con}(BAG)$$

and then working in W we construct a generic extension which satisfies statements of our interest. This will give us relative consistency results.

11.2. Definition Assume $\mathbb{P} \in M$ is a poset, G is a filter generic for $\mathbb{P}$ over M and $x \in M$ we define the evaluation (x^G) of x by G as follows:

$$x^G = \{ y^G \mid (\exists p \in G)(\langle p, y \rangle \in x) \}$$

This is understood as definition by recursion on $\in$.

11.3. Remark Each set is associated with its characteristic function. If $A \in M$ and $Z \subseteq A$ then the the characteristic function $\chi_Z : A \longrightarrow \{0,1\}$ carries the same amount of info as Z.

Think of $\mathbb{P} = \{0,1\}$ with $0 < 1$ and $G = \{1\}$. Then

$$(\chi_Z^{-1})^G = \chi_Z^{-1}[G] = Z$$

Now more generally, we have poset $\mathbb{P}$ and a generic $G \notin M$. If $x : A \longrightarrow \mathbb{P}$ then $(x^{-1})^G = x^{-1}[G]$. Now we can have $(x^{-1})^G \notin M$.

But in general, if $\mathbb{P}$ is a poset asking x to be a function may be too restrictive, in the sense this way we may not be able to generate all sets of interests. Therefore we allow x to be a binary relation with $\mathrm{dom}(x) \subseteq \mathbb{P}$.

Notice the above only treats the case of subsets of $A \in M$. In order to build a model with G in it we also need to add sets with elements of the form $(x^{-1})^G$ which are not in M and go up the cumulative hierarchy. Notice Def 11.2. does this.

When working with evaluations, it will be often useful to consider x we are evaluating of certain specific form.

11.4. Definition Let $\mathbb{P}$ be a poset. We say that x is a $\mathbb{P}$-term ($\mathbb{P}$-name) iff

Elements of x are ordered pairs $\langle p, y\rangle$ s.t. $p \in \mathbb{P}$ and y is a $\mathbb{P}$-term

This is a definition by recursion on $\in$. (Exercise: Write a formally correct form of this def.).

If M is a model as above, $\mathbb{P} \in M$ is a poset we denote

$M^{\mathbb{P}}$ = the class of all $\mathbb{P}$-names in M

11.5. Proposition Assume M is a transitive model, $\mathbb{P} \in M$ is a poset and $x \in M$. Then there is a $\mathbb{P}$-term $\check{x}$ such that for every filter G generic for $\mathbb{P}$ over M:

$$x^G = \check{x}^G$$

Proof Exercise (Remove all $z \in \mathrm{trcl}(x)$ which are not ordered pairs of the right kind.)

11.6. Terminology Instead of saying "G is a filter generic for $\mathbb{P}$ over M" we say "G is a $(\mathbb{P}, M)$-generic filter".

11.7. Definition (Canonical terms) Let $\mathbb{P} \in M$ be a poset.

(a) For $a \in M$ we let

$$\check{a} = \{ \langle 1_{\mathbb{P}}, \check{x} \rangle \mid x \in a \}$$

(b) $\dot{G} = \{ \langle p, \check{p} \rangle \mid p \in \mathbb{P} \}$

11.8. Proposition For every $(\mathbb{P}, M)$ generic G:

(a) $\check{a}^G = a$ (For this reason $\check{a}$ is called the "check name for a".)

(b) $\dot{G}^G = G$ (So if H is $(\mathbb{P}, M)$-generic then $\dot{G}^H = H$)

Proof Exercise.

11.9. Proposition Let G be $(\mathbb{P}, M)$-generic and $x \in M$. Then $\operatorname{rank}(x^G) \leq \operatorname{rank}(x)$.

Proof Exercise (Recursion on $\in$)

11.10. Definition Let G be $(\mathbb{P}, M)$-generic. We define

$$M[G] = \{ x^G \mid x \in M \}$$

$M[G]$ is called the generic extension of M by G.

11.11. Proposition Let G be $(\mathbb{P}, M)$-generic. Then

(a) $M \subseteq M[G]$ and $G \in M[G]$

(b) If $N \models ZF$ is transitive such that $M \subseteq N$ and $G \in N$ then $M[G] \subseteq N$

(c) $M[G]$ is transitive

(d) $On^{M[G]} = On^M$

Proof (a) If $x \in M$ then $x = \check{x}^G \in M[G]$. Also $G = \dot{G}^G \in M[G]$

(b) Immediate from the definitions of $M[G]$ and that of evaluation. If $x, G \in N$ then $x^G \in N$.

(c) Immediate from the def of evaluation: if $z \in x^G$ then $z = y^G$ for some $y \in M$. So $z \in M[G]$

(d) $On^M \subseteq On^{M[G]}$ as $\alpha = \check{\alpha}^G$ for all $\alpha \in On$

To see $On^{M[G]} \subseteq On^M$: if $\alpha \in On^{M[G]}$ then $\alpha = x^G$ for some $x \in M$.

But by P 11.9. $\alpha = \mathrm{rank}(\alpha) = \mathrm{rank}(x^G) \leq \mathrm{rank}(x) \in On^M$. $\square$(P 11.11)

11.12. Theorem (Forcing theorem) There is a recursive assignment of formulae in LST

$$\varphi(v_1, \dots v_e) \longmapsto \varphi^*(u, u', v_1 \dots v_e)$$

such that the following holds

(A) If M is a transitive model, $\mathbb{P} \in M$ is a poset, $p \in \mathbb{P}$ and $x_1, \dots, x_e \in M$ then

$$M \models \varphi^*(\mathbb{P}, p, x_1 \dots x_e)$$

iff

For every $(\mathbb{P}, M)$-generic filter G s.t. $p \in G$:

$$M[G] \models \varphi(x_1^G, \dots, x_e^G)$$

(B) If G is a $(\mathbb{P}, M)$-generic filter and $x_1 \dots x_e \in M$ are such that

$$M[G] \models \varphi(x_1^G, \dots, x_e^G)$$

then there is a condition $p \in G$ such that

$$M \models \varphi^*(\mathbb{P}, p, x_1 \dots, x_e)$$

11.13. Remarks

(a) By "model" in Th 11.12 we mean a model of sufficient fragment of ZF or ZFC. ZF or ZFC itself is usually safe.

(b) By "every generic" or "some generic" we mean "every / some" generic in V.

(c) Instead of

$$M \models \varphi^*(\mathbb{P}, p, x_1 \dots x_e)$$

we write
$$p \Vdash^{M}_{\mathbb{P}} \varphi(x_1 \ldots x_e)$$
and read "p forces φ at $x_1 \ldots x_e$ over M".
If M, $\mathbb{P}$ are clear from the context we write

$p \Vdash_{\mathbb{P}} \varphi(x_1 \ldots x_e)$ or $p \Vdash \varphi(x_1 \ldots x_e)$

(d) $\Vdash$ is called the <u>forcing relation</u>. But notice this is a schema, for each formula φ we have one relation. We cannot define all of them by Tarski's Thm.
If $\mathbb{P} = \{0, 1\}$ with $0 < 1$ then $G \in M$ for any $(\mathbb{P}, M)$-generic G, & $M[G] = M$. Easy to check:
$$p \Vdash^{M}_{\mathbb{P}} \varphi(\check{x}_1, \ldots, \check{x}_e) \quad \text{iff} \quad M \models \varphi(x_1 \ldots x_e)$$
So if we could define $\Vdash^{M}_{\mathbb{P}}$ by a single formula, we could also define $\models$ this way.