11. Basics of Forcing

11.1. Background setting. We will work with a model $M$ of a sufficiently large fragment of $\text{ZFC}$. We will often have the full $\text{ZFC}$ or $\text{ZFC}$. Typically, $M$ will be an universe, so we will often write $V$ for $M$. $M$ will be the model we will force over; i.e. we will have a poset $P \subseteq M$ and a filter $G$ generic for $P$ over $M$. (In our previous terminology: $G$ is $P$-generic when $G = \{ D \in P : D \text{ is dense in } P \}$.)

and we will "adjoin" $G$ to $M$. In order that this can be done correctly, we will make the following assumption: "Background assumption on generics."

(BAG) We have a model $W$ of $\text{ZFC}$ such that:

1. $M$ is a transitive structure in $W$.
2. For every $p \in P$ there is a filter $G \in W$ generic for $P$ over $M$ such that $p \in G$.

Point: One can show that it is consistent to have this situation, assuming $\text{ZFC}$ is consistent. So we prove a meta-theorem of the form:

$$\text{Con}(\text{ZFC}) \Rightarrow \text{Con}(\text{BAG})$$

and then working in $W$ we construct a generic extension which satisfies statements of our interest. This will give us relative consistency results.

11.2. Defining. Assume $P \subseteq M$ is a poset, $G$ is a filter generic for $P$ over $M$ and $x \in M$ we define the evaluation of $x$ by $G$ as follows:

$$x^G = \{ y : (\exists p \in G)( (p, y) \in x) \}$$

This is understood as definition by recursion on $e$. 
11.3. Remark. Each set is associated with the characteristic function. If \( A \subseteq M \) and \( \mathcal{Z} \subseteq A \) then the characteristic function \( f_2 : A \to \{0,1\} \) carries the same amount of info as \( \mathcal{Z} \).

Think of \( \mathcal{P} = \{0,1\} \) with \( 0 \leq 1 \) and \( \mathcal{E} = \{1\} \). Then
\[
(\mathcal{E}^* \frieden) = f_2^* [\mathcal{E}] = 2
\]

Now more generally, we have a point \( \mathcal{P} \) and a generic \( \mathcal{G} \) in \( \mathcal{M} \).
If \( x : A \to \mathcal{P} \) then \( (x)^* = x^* [\mathcal{G}] \). Now we can build \( \mathcal{E}^\mathcal{G} \) but in general, if \( \mathcal{P} \) is a point asking \( x \) to be a function may be too restrictive, in the sense that we may not be able to generate all sorts of interests. Therefore we allow \( x \) to be a binary relation with \( \text{dom}(x) \subseteq \mathcal{P} \).

Notice the above and hearken the case of subsets of \( A \subseteq M \).
In order to build a model with \( \mathcal{E} \) in it we also need to add sets with elements of the form \( \mathcal{E}^\mathcal{G} \) which are not in \( \mathcal{M} \) and go up the cumulative hierarchy. Notice Def 11.2.

When working with evaluations, it will be often useful to consider \( x \) one equivalence of certain specific form.

11.4. Definition. Let \( \mathcal{P} \) be a point. We say that \( x \) is a \( \mathcal{P} \)-term (\( \mathcal{P} \)-name) if

\[
\text{Elements of } x \text{ are ordered pairs } \langle p , y \rangle \text{ s.t. } p \in \mathcal{P} \text{ and } y \text{ is a } \mathcal{P} \text{-term.}
\]

This is a definition by recursion on \( \mathcal{E} \). Exercise: Write a truly correct form of this def. 1.

If \( \mathcal{M} \) is a model as above, \( \mathcal{P} \subseteq \mathcal{M} \) is a poset we denote
\( \mathcal{P}^\mathcal{M} = \text{the class of all } \mathcal{P} \text{-names in } \mathcal{M} \).

11.5. Proposition. Assume \( \mathcal{M} \) is a transitive model, \( \mathcal{P} \subseteq \mathcal{M} \text{ is a poset and } x \in \mathcal{M} \). Then there is a \( \mathcal{P} \)-term \( x \) such that for every generic \( \mathcal{G} \) generic for \( \mathcal{P} \) over \( \mathcal{M} \):
\[
x^\mathcal{G} = x^\mathcal{G}
\]

Proof. Exercise. (Remove all \( x \in \mathcal{E} \) and \( (x) \) which are not
ordered pairs of the right kind.

11.6. Terminology. Instead of saying "$G$ is a filter generic for $P$ over $M$" we say "$G$ is a $(P, M)$-generic filter".

11.7. Definition. (Canonical terms) Let $P \in M$ be a poset.

(a) For $a \in M$ we let
   $$\check{a} = \{ < \check{p}, \check{x} > \mid x \in a \}$$

(b) $\dot{G} = \{ < \check{p}, \check{x} > \mid p \in P \}$

11.8. Proposition. For every $(P, M)$ generic $G$:
   
   (a) $\check{a}^G = a$ (For this reason $\check{a}$ is called the "check name for $a$".)

   (b) $\dot{G}^G = G$ (So if $H$ is $(P, M)$-generic then $\check{G}^H = H$

Proof. Exercise.

11.9. Proposition. Let $G$ be $(P, M)$-generic and $x \in M$.

Then $\text{rank}(\check{x}^G) \leq \text{rank}(x)$.

Proof. Exercise (Recursion on $\leq$)

11.10. Definition. Let $G$ be $(P, M)$-generic. We define

$M[G] = \{ x^G \mid x \in M \}$

$M[G]$ is called the generic extension of $M$ by $G$.

11.11. Proposition. Let $G$ be $(P, M)$-generic. Then

(a) $M \subseteq M[G]$ and $G \in M[G]

(b) If $N \not\subseteq F$ is transitive such that $M \subseteq N$ and $G \in N$ then $M[G] \subseteq N$

(c) $M[G]$ is transitive

(d) $\text{On}^M[G] = \text{On}^M$

Proof. (a) If $x \in M$ then $x = x^G \in M[G]$. Also $G = \check{G}^G \in M[G]$

(b) Immediate from the definitions of $M[G]$ and that of evaluation.

(c) Immediate from the definition of evaluation: if $z \in x^G$ then $z = G$

for some $y \in M$. So $z \in M[G]$.
11.12. Theorem (Forcing theorem) There is a recursive assignment of formulas in $L^T$

$$\varphi(v_1, \ldots, v_k) \mapsto \varphi^*(u_1, \ldots, u_k)$$

such that the following holds:

(A) If $M$ is a transitive model, $P \in M$ is a poset, $p \in P$, and $x_1, \ldots, x_k \in M$ then

$$M \models \varphi^*(P, p, x_1, \ldots, x_k)$$

if and only if

For every $(P, M)$-generic filter $G$ such that $p \in G$:

$$M[G] \models \varphi(x_1^G, \ldots, x_k^G)$$

(B) If $G$ is a $(P, M)$-generic filter and $x_1, \ldots, x_k \in M$ are

such that

$$M[G] \models \varphi(x_1^G, \ldots, x_k^G)$$

then there is a condition $p \in G$ such that

$$M \models \varphi^*(P, p, x_1, \ldots, x_k)$$

11.13. Remarks

(a) By "model" in THM. 12 we mean a model of sufficient fragment of $ZFC$ itself or usual $ZFC$.

(b) By "any generic" or "some generic" we mean "every/some" generic in $M$.

(c) Instead of

$$M \models \varphi^*(P, p, x_1, \ldots, x_k)$$
we write

\[ p \vdash_M \varphi(x_1, \ldots, x_n) \]

and read "p forces \( \varphi \) at \( x_1, \ldots, x_n \) over \( M \)".

If \( M, P \) are clear from the context we write

\[ p \vdash \varphi(x_1, \ldots, x_n) \quad \text{or} \quad p \vdash \varphi(x, \ldots, x) \]

(d) It is called the forcing relation. But notice this is a schema, for each bounded we have one relation. We cannot define all of them by Tarski's Theorem. If \( P = \{0, 1\} \) with \( \prec \subset \) then \( G \in M \) for every involutive \( \varphi \)-generic \( G \), so \( M[G] = M \). E.g. to check:

\[ p \vdash_M \varphi(x_1, \ldots, x_n) \quad \text{iff} \quad M \models \varphi(x_1, \ldots, x_n) \]

So if we could define \( \vdash^M \) by a single formula, we could also define \( \models \) this way.