Theorem (Generic Extension Theorem) Assume $M \models ZF$, $P \in M$ is a poset and $G$ is a $(P, M)$-generic filter. Then $M[G] \models ZF$. Moreover, if $M \models ZFC$ then $M[G] \models ZFC$.

The proof of Theorem actually shows the following.

Theorem Assume $\sigma_1, \ldots, \sigma_n$ are axioms of $ZFC$. Then there are axioms of $ZFC$ $\tau_1, \ldots, \tau_k$ such that if $P \in M$ is a poset, $G \in (P, M)$-generic and

$$M \models \{\sigma_1, \ldots, \sigma_n\} \cup \{\tau_1, \ldots, \tau_k\}$$

Then $M[G] \models \{\sigma_1, \ldots, \sigma_n\}$.

Proposition (Basic properties of the forcing relation)

1. If $p \models \varphi(x_1, \ldots, x_e)$ and $q \leq p$ then $q \models \varphi(x_1, \ldots, x_e)$
2. $p \models \varphi(x_1, \ldots, x_e)$ if and only if $q \models \varphi(x_1, \ldots, x_e)$ for all $q \leq p$
3. $p \not\models \varphi(x_1, \ldots, x_e)$ if and only if $q \not\models \varphi(x_1, \ldots, x_e)$ for some $q \leq p$
4. $p \models (\forall x \in X) \varphi(x_1, \ldots, x_e)$ if and only if $p \models q \models \varphi(x_1, \ldots, x_e)$ for all $q \leq p$
5. $p \models (\exists x \in X) \varphi(x_1, \ldots, x_e)$ if and only if $q \models \varphi(x_1, \ldots, x_e)$ for all $q \leq p$
6. $p \models (\forall x \in X) \varphi(x_1, \ldots, x_e)$ if and only if $q \models \varphi(x_1, \ldots, x_e)$ for all $q \leq p$

Logical rules

(MP) $p \models \varphi(x_1, \ldots, x_e)$ and $p \models (\varphi \rightarrow \psi)(x_1, \ldots, x_e)$ then $p \models \psi(x_1, \ldots, x_e)$

(QR) Let $\varphi(v, x_1, \ldots, x_e)$ be a formula such that $v$ has no free occurrence in $p$.

If $p \models \varphi(x_1, \ldots, x_e)$ for all $x$ then $p \models (\exists v) \varphi(v, x_1, \ldots, x_e)$.
Axioms of predicate logic

(APL) If $\varphi(\bar{x}_1, \ldots, \bar{x}_e)$ is an axiom of predicate logic then

$$p \vdash \varphi(\bar{x}_1, \ldots, \bar{x}_e) \quad \text{for all } \bar{x}_1, \ldots, \bar{x}_e.$$

So we get: Let $\sigma$ be a sentence in LST.

If $2E \vdash \sigma$ then $p \vdash \sigma$.

Proof

(1) Pick $\sigma \in (P, M)$ - quod G s.t. $q \in G$. Since $q \in p$ we have $p \in G$. Since $p \vdash \varphi(\bar{x}_1, \ldots, \bar{x}_e)$, we have

$$\mathcal{M}(\sigma) \models \varphi(\bar{x}_1, \ldots, \bar{x}_e).$$

That is:

$$q \in G \implies \mathcal{M}(\sigma) \models \varphi(\bar{x}_1, \ldots, \bar{x}_e).$$

So by the Four-Valued Principle $p \vdash \varphi(\bar{x}_1, \ldots, \bar{x}_e)$.

(2) The non-trivial part is: If

$$D = \{ q \in p \mid q \vdash \varphi(\bar{x}_1, \ldots, \bar{x}_e) \}$$

is dense below $p$ then $p \vdash \varphi(\bar{x}_1, \ldots, \bar{x}_e)$

Assume $G \ni p$ is $(P, M)$ - generic. Prove: prove that $G \cap D \neq \emptyset$. (Exercise)

(3) $\subseteq$ Assume $q \in p$ and $q \vdash \varphi(\bar{x}_1, \ldots, \bar{x}_e)$. Pick $G \ni q$, which is $(P, M)$ - generic. Then $\mathcal{M}(\sigma) \models \varphi(\bar{x}_1, \ldots, \bar{x}_e)$ and $p \in G$ as $q \in p$. So $p \vdash \varphi(\bar{x}_1, \ldots, \bar{x}_e)$

$\supseteq$ Now assume $p \vdash \varphi(\bar{x}_1, \ldots, \bar{x}_e)$. So we can find a $(P, M)$ - generic $G$ s.t. $p \in G$ and $\mathcal{M}(\sigma) \models \varphi(\bar{x}_1, \ldots, \bar{x}_e)$. By (B) on the Four-Valued Principle we can find $p \in G$ such that $p \vdash \varphi(\bar{x}_1, \ldots, \bar{x}_e)$. Because $p \vdash \varphi(\bar{x}_1, \ldots, \bar{x}_e)$. Because $p \vdash \varphi(\bar{x}_1, \ldots, \bar{x}_e)$, but we don't need it here.) s.t. $q \in p$. Then

$q \in p$ and $q \vdash \varphi(\bar{x}_1, \ldots, \bar{x}_e)$.

(4) Exercise
The non-trivial part is following: if
\[ D = \{ q \in p \mid q \not\in \neg \varphi(x_1, \ldots, x_n) \} \]
\[ \text{is dense below } p \text{ then } p \Vdash \neg \varphi(x_1, \ldots, x_n). \]
To see this, using (3) we show
\[ D' = \{ q \leq p \mid q \not\in \neg \varphi(x_1, \ldots, x_n) \} \]
\[ \text{is dense below } p. \text{ Then use the Forcing Theorem for the fact that } D \cup G \neq \emptyset \text{ for any } (p, M) \text{-generic } \mathcal{G}, \]
s.t. \( p \in G \).

Let us do \( \Rightarrow \). Assume \( p \Vdash (\exists x) \varphi(x_1, x_2, \ldots, x_n) \). Let \( q \leq p \) and \( G \) be a \((p, M)\)-generic s.t. \( q \in G \). By Forcing Theorem Claim (A): \( M[G] \models (\exists x) \varphi(x_1, x_2, \ldots, x_n) \). So we have some \( a \in M[G] \) s.t. \( M[G] \models \varphi(a, x_2, \ldots, x_n) \). By the definition of \( M[G] \):
\[ a = \check{y} \text{ for some } y \in M, \text{ so } M[G] \models \varphi(y, x_2, \ldots, x_n) \]. By Forcing Theorem Claim (B) then \( q \in G \) s.t. \( q \Vdash \neg \varphi(y, x_1, \ldots, x_n) \).

Now since \( q, q' \in G \) we can find \( r \in q, q' \); then
\[ \forall \alpha \leq r \varphi(x_1, y, \ldots, x_n) \] and \( r \in p \).

Exercise. \( \Box \) P. 44. 16.

11. 17. Proposition. Basic examples of posets. Given a cardinal \( \kappa \) and sets \( A, B \) s.t. \( A \) is well-orderable, let
\[ \text{FN}(A, B, \kappa) = \text{the poset of all functions } p : A \to B \text{ s.t. } \in A \text{ is ordered by reverse inclusion.} \]

We show have the following points.

For a regular cardinal \( \kappa \) and any set \( A : \text{col} \kappa (\kappa, A) = \text{FN}(\kappa, A, \kappa) \)
This is called the collapse forcing, by a density argument of \( G \) is \( (\text{col} \kappa (\kappa, A), M) \)-generic and \( q = U_G \) then
\[ q : \kappa \to A \text{ is a surjection (Exercise).} \]
In particular: if \( \kappa \) is a cardinal in \( M \) then \( \kappa \) is any generic extension by \( \text{col} \kappa (\kappa, \kappa) \), \( \kappa \) has
For cardinals \( \kappa, \lambda \):

\[ \text{Add}(\kappa, \lambda) = \text{Fn}(\kappa \times \lambda, \{0, 1\}, \kappa) \]

If \( \tau \in \text{Add}(\kappa, \lambda) \) is \( \kappa \)-generic and \( \varphi = \text{UG} \) then by a density argument

\[ \varphi : \kappa \times \lambda \rightarrow \{0, 1\} \]

such that if we let \( \varphi_\lambda : \kappa \rightarrow \{0, 1\} \) be defined by

\[ \varphi_\lambda(\iota) = \varphi(\iota, \lambda) \quad \text{when} \ \iota < \kappa \]

then \( \varphi \neq \varphi_\lambda \), and hence, letting

\[ a_\varphi = \{ \iota < \kappa \mid \varphi_\lambda(\iota) = \lambda \} \]

we have

\[ \tau \neq \beta \Rightarrow \varphi_\lambda(\tau) \neq \varphi_\lambda(\beta) \]

So we have \( \lambda \) many distinct subsets of \( \kappa \) in \( M^{\varphi_\lambda} \).

An \( \lambda \)-subset of \( \kappa \) added to a model by \( \text{Fn}(\kappa, \lambda, \{0, 1\}, \kappa) \)

is called a Cohen subset of \( \kappa \). So \( \text{Add}(\kappa, \lambda) \) adds \( \lambda \) many Cohen subsets of \( \kappa \) to \( M \).

Given a regular cardinal \( \kappa \) and a set \( A \):

\[ \text{Coll}(\kappa, A) = \text{the set of all functions } \varphi \text{ s.t.} \]

\[ \cdot \text{dom}(\varphi) \subseteq [A \times \kappa]^{< \kappa} \]

\[ \cdot \text{For any } (a, \beta) \in \text{dom}(\varphi): \]

\[ \varphi(a, \beta) \in A \]

By a density argument of \( G \in \text{Coll}(\kappa, A), \kappa \)-generic and \( \varphi = \text{UG} \) then

\[ \varphi : \kappa \times \kappa \rightarrow UA \]

and for every \( a \subseteq A \) the function \( \varphi_a : \kappa \rightarrow a \) defined by

\[ \varphi_a(\beta) = \varphi(a, \beta) \]

is a surjection of \( \kappa \) onto \( a \). So \( \text{Coll}(\kappa, A) \)
adds a surjection of \( \kappa \) onto \( a \) for every \( a \subseteq A \).
The poset \( \text{Coll}(\alpha, \kappa) \) is called the Levy collapse of \( \text{Coll}(\alpha) \) to \( \kappa \).

If \( \kappa \in \Omega \), then we usually write \( \text{Coll}(\alpha, < \kappa) \) instead of \( \text{Coll}(\alpha, \kappa) \).

1.11. Lemma. Work in \( \mathbb{M}_{\mathcal{E} \mathcal{C}} \). Assume \( \mathcal{P} \) a \( \kappa \)-c.c. poset when \( \kappa \) is a cardinal. Let \( A, B \in \mathcal{M}, \ G \in \mathcal{P} \) be \( (\mathcal{P}, \mathcal{M}) \)-generic and \( f : A \rightarrow B \) be s.t. \( f \in \mathcal{M}(G) \).

Then there is a function \( F \in \mathcal{M} \) s.t.

- (i) \( \text{dom}(F) = A \)
- (ii) \( F(a) \in F(a) \subseteq B \) for all \( a \in A \)
- (iii) \( \text{card}^\mathcal{M}(F(a)) < \kappa \) for all \( a \in A \)

\[ \text{Proof:} \quad \text{Since } f : A \rightarrow B \text{ in } \mathcal{M}(G), \text{ by the definition of } \mathcal{M}(G) \text{ there is some } \mathcal{P}-\text{term } \tilde{f} \in \mathcal{M} \text{ s.t. } f = \tilde{f} \mathcal{G}. \text{ So } \]

\[ \mathcal{M}(G) \models \tilde{f} : A \rightarrow B \]

By Forcing Theorem Clause (2) there is some \( p \in G \) s.t.

\[ \text{p} \vdash \tilde{f} : A \rightarrow B \]

Fix \( a \in A \).

(i) \( b \neq b' \) are elements of \( B \) and \( a \neq a' \) are conditions below \( \mathcal{P} \) s.t.

\[ \text{p} \vdash \tilde{f}(a) = b \quad \text{and} \quad \text{p} \vdash \tilde{f}(a') = b' \]

then \( a \perp a' \).

Why? If not, there would be some \( c \leq a, a' \) that \( f \nabla \tilde{f}(a) = f \nabla \tilde{f}(a') \). Now if
$H$ is any $(P, \mathcal{H})$-generic with $\alpha \in H$ then

$q, q' \in H$, hence

\[ \hat{f}^H(\alpha^H) = b^H \quad \hat{f}^H(\alpha^H) = b^H \]

\[ \hat{f}^H(a) = b \quad \hat{f}^H(a) = b' \]

Technically: $<q, b> \in \hat{f}^H$  

But recall $\hat{f}^H$ is a function, and $p \in H$  

becomes $a \leq q, q' \leq p$. Contradiction. \( \square \).

Now assume $a \in A$ let

\[ D_a = \{ q \in p \mid (\exists b \in B)(q \vdash \hat{f}(a) = b) \} \]

Exercise $D_a$ is an open subset of $p^A$. 