M281C SPRING 2020 L9

11.14. Theorem (Generic extensions Thm) Assume $M \models ZF$, $\mathbb{P} \in M$ is a poset and G is a $(\mathbb{P}, M)$-generic filter. Then $M[G] \models ZF$. Moreover if $M \models ZFC$ then $M[G] \models ZFC$.

The proof of T 11.14. actually shows the following.

11.15. Theorem Assume $\sigma_1, \dots, \sigma_n$ are axioms of ZFC. Then there are axioms of ZFC $\tau_1, \dots, \tau_k$ such that if $\mathbb{P} \in M$ is a poset, G is $(\mathbb{P}, M)$-generic and

$$M \models \{\sigma_1, \dots \sigma_n\} \cup \{\tau_1 \dots \tau_k\}$$

Then $M[G] \models \{\sigma_1, \dots, \sigma_n\}$.

11.16. Proposition (Basic properties of the forcing relation)

① if $p \Vdash \varphi(x_1, \dots x_\ell)$ and $q \le p$ then $q \Vdash \varphi(x_1, \dots x_\ell)$

② $p \Vdash \varphi(x_1 \dots x_\ell)$ iff $q \Vdash \varphi(x_1 \dots x_\ell)$ for all $q \le p$
iff $q \Vdash \varphi(x_1 \dots x_\ell)$ for densely many $q \le p$.

③ $p \not\Vdash \varphi(x_1 \dots x_\ell)$ iff $q \Vdash \neg\varphi(x_1 \dots x_\ell)$ for some $q \le p$

④ $p \Vdash (\varphi \wedge \psi)(x_1 \dots x_\ell)$ iff $\left(p \Vdash \varphi(x_1 \dots x_\ell) \text{ and } p \Vdash \psi(x_1 \dots x_\ell)\right)$

⑤ $p \Vdash \neg\varphi(x_1 \dots x_\ell)$ iff $q \not\Vdash \varphi(x_1 \dots x_\ell)$ for all $q \le p$
iff $q \not\Vdash \varphi(x_1 \dots x_\ell)$ for densely many $q \le p$

⑥ $p \Vdash (\exists v)\varphi(v, x_1 \dots x_\ell)$ iff for densely many $q \le p$ there is y with $q \Vdash \varphi(y, x_1 \dots x_\ell)$

⑦ Logical rules

(MP) $p \Vdash \varphi(x_1 \dots x_\ell)$ and $p \Vdash (\varphi \to \psi)(x_1 \dots x_\ell)$ then $p \Vdash \psi(x_1 \dots x_\ell)$

(QR) Let $\varphi(v, v_1 \dots v_\ell)$, $\psi(v_1 \dots v_\ell)$ be formulae s.t. v has no free occurrence in ψ.
If $p \Vdash \varphi(x, x_1 \dots x_\ell) \to \psi(x_1 \dots x_\ell)$ for all x then $p \Vdash (\exists v)\varphi(v, x_1 \dots x_\ell)$ |

Axioms of Predicate logic

(APL) if $\varphi(v_1 \ldots v_e)$ is an axiom of predicate logic then
$$p \Vdash \varphi(x_1 \ldots x_e) \quad \text{for all } x_1 \ldots x_e.$$

So we get: Let σ be a sentence in LST.

If $ZF \vdash \sigma$ then $p \Vdash \sigma$

Proof ① Pick a $(\mathbb{P}, M)$-generic G s.t. $q \in G$. Since $q \leq p$ we have $p \in G$. Since $p \Vdash \varphi(x_1 \ldots x_e)$, we have
$$M[G] \models \varphi(x_1^G \ldots x_e^G)$$
That is:
$$q \in G \implies M[G] \models \varphi(x_1^G, \ldots, x_e^G)$$
So by the Forcing Thm $q \Vdash \varphi(x_1 \ldots x_e)$.

② The non-trivial part is: if
$$D = \{ q \leq p \mid q \Vdash \varphi(x_1 \ldots x_e) \}$$
is dense below p then $p \Vdash \varphi(x_1 \ldots x_e)$
Assume $G \ni p$ is $(\mathbb{P}, M)$-generic. Point: prove that $G \cap D \neq \emptyset$. (Exercise)

③ $\Leftarrow$ Assume $q \leq p$ and $q \Vdash \neg\varphi(x_1 \ldots x_e)$. Pick $G \ni q$ which is $(\mathbb{P}, M)$-generic. Then $M[G] \models \neg\varphi(x_1^G \ldots x_e^G)$, and $p \in G$ as $q \leq p$. So $p \nVdash \varphi(x_1 \ldots x_e)$

$\Rightarrow$ Now assume $p \nVdash \varphi(x_1 \ldots x_e)$. So we can find a $(\mathbb{P}, M)$-generic G s.t. $p \in G$ and $M[G] \models \neg\varphi(x_1 \ldots x_e)$. By (B) in the Forcing Thm we can find $p' \in G$ such that $p' \Vdash \neg\varphi(x_1 \ldots x_e)$. Because $p, p' \in G$ we can find q (in G, but we don't need it here) s.t. $q \leq p, p'$. Then $q \leq p$ and $q \Vdash \neg\varphi(x_1 \ldots x_e)$.

④ Exercise

(5) The non-trivial part is following: If

$$D = \{ q \leq p \mid q \not\Vdash \varphi(x_1 .. x_e) \}$$

is dense below p then $p \Vdash \neg\varphi(x_1 .. x_e)$.

To see this, using (3) we show

$$D' = \{ q \leq p \mid q \Vdash \neg\varphi(x_1 .. x_e) \}$$

is dense below p. Then use the Forcing Thm plus the fact that $D' \cap G \neq \emptyset$ for any $(\mathbb{P}, M)$-generic G s.t. $p \in G$

(6) Let us do $\Rightarrow$. Assume $p \Vdash (\exists v)\varphi(v, x_1 .. x_e)$. Let $q \leq p$ and G be a $(\mathbb{P}, M)$-generic s.t. $q \in G$. By Forcing Thm clause (A): $M[G] \models (\exists v)\varphi(v, x_1^G ... x_e^G)$. So we have some $a \in M[G]$ s.t. $M[G] \models \varphi(a, x_1^G ... x_e^G)$. By the definition of $M[G]$: $a = y^G$ for some $y \in M$, so $M[G] \models \varphi(y^G, x_1^G ... x_e^G)$. By Forcing Thm Clause (B) there is some $q' \in G$ s.t. $q' \Vdash \varphi(y, x_1 ... x_e)$. Now since $q, q' \in G$ we can find $r \leq q, q'$; then $r \Vdash \varphi(y, x_1 .. x_e)$ and $r \leq q$.

(7) Exercise. $\square$ P 11.16.

<u>11.17. Proposition</u> Basic examples of posets. Given a cardinal κ and sets A, B s.t. A is well-ordarable, let

$FN(A, B, \kappa)$ = the poset of all functions $p : a \to B$ s.t. $a \in ([A]^{<\kappa})$ ordered by reverse inclusion

We then have the following posets

(a) For a regular cardinal κ and any set A:

$coll(\kappa, A) = FN(\kappa, A, \kappa)$

This is called the <u>collapse forcing</u>, by a density argument if G is $(coll(\kappa, A), M)$-generic and $g = \bigcup G$ then $g : \kappa \to A$ is a surjection (Exercise)

In particular: If λ is a cardinal in M then in any generic extension by $coll(\kappa, \lambda)$, λ has cardinality $\leq \kappa$.

(b) For cardinals κ, λ:

$\mathrm{Add}(\kappa, \lambda) = \mathrm{Fn}(\lambda \times \kappa, \{0,1\}, \kappa)$

if G is $(\mathrm{Add}(\kappa,\lambda), M)$-generic and $g = \bigcup G$ then by a density argument

$g : \lambda \times \kappa \to \{0,1\}$.

such that if we let $g_\alpha : \kappa \to \{0,1\}$ be defined by

$g_\alpha(\xi) = g(\alpha, \xi)$ when $\alpha < \lambda$

then $\alpha \neq \beta \Rightarrow g_\alpha \neq g_\beta$, and hence, letting

$a_\alpha = \{\xi < \kappa \mid g_\alpha(\xi) = 1\}$

we have

$\alpha \neq \beta \Rightarrow a_\alpha \neq a_\beta$

So we have λ many distinct subsets of κ in $M[G]$.

A subset of κ added to a model by $\mathrm{Fn}(\kappa, \{0,1\}, \kappa)$ "=" $\mathrm{Add}(\kappa, 1)$ is called a Cohen subset of κ. So $\mathrm{Add}(\kappa, \lambda)$ adds λ many Cohen subsets of κ to M.

(c) Given a regular cardinal κ and a set A:

$\mathrm{Coll}(\kappa, A)$ = the set of all functions p s.t.

- $\mathrm{dom}(p) \in [A \times \kappa]^{<\kappa}$
- For every $(a, \xi) \in \mathrm{dom}(p)$:

$p(a, \xi) \in a$

By a density argument if G is $(\mathrm{Coll}(\kappa, A), M)$-generic and $g = \bigcup G$ then

$g : A \times \kappa \longrightarrow \bigcup A$

and for every $a \in A$ the function $g_a : \kappa \longrightarrow a$ defined by

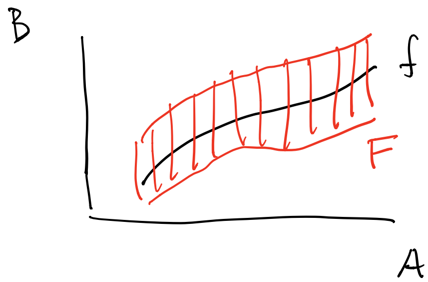
$g_a(\xi) = g(a, \xi)$

is a surjection of κ onto a. So $\mathrm{Coll}(\kappa, A)$ adds a surjection of κ onto a for every $a \in A$.

The poset $\mathrm{Coll}(\kappa, A)$ is called the Lévy collapse of (elements of) A to κ.
If $\lambda \in \mathrm{On}$ then we usually write $\mathrm{Coll}(\kappa, <\lambda)$ instead of $\mathrm{Coll}(\kappa, \lambda)$.

<u>11.18. Lemma</u> Work in $M \models ZFC$. Assume $\mathbb{P}$ a κ-c.c. poset where κ is a cardinal. Let $A, B \in M$, G be $(\mathbb{P}, M)$-generic and $f: A \to B$ be s.t. $f \in M[G]$.
Then there is a function $F \in M$ s.t.

(i) $\mathrm{dom}(F) = A$
(ii) $f(a) \in F(a) \subseteq B$ all $a \in A$
(iii) $\mathrm{card}^M(F(a)) < \kappa$ all $a \in A$



<u>Proof</u> Since $f: A \to B$ in $M[G]$, by the definition of $M[G]$ there is some $\mathbb{P}$-term $\dot{f} \in M$ s.t. $f = \dot{f}^G$. So
$$M[G] \models \dot{f}^G : \check{A}^G \longrightarrow \check{B}^G$$
By Forcing Thm Clause (A) there is some $p \in G$ s.t.
$$p \Vdash \dot{f} : \check{A} \longrightarrow \check{B}$$
Fix $a \in A$.

(1) If $b \neq b'$ are elements of B and $q \neq q'$ are conditions below p s.t.
$$q \Vdash \dot{f}(\check{a}) = \check{b} \quad \text{and} \quad q' \Vdash \dot{f}(\check{a}) = \check{b}'$$
then $q \perp q'$.

<u>Why</u>: If not, there would be some $r \leq q, q'$. Now if

H is any $(\mathbb{P}, M)$-generic with $r \in H$ then $q, q' \in H$, hence

$$\dot{f}^H(\check{a}^H) = \check{b}^H \qquad \dot{f}^H(\check{a}^H) = \check{b}'^H$$

$$\dot{f}^H(a) = b \qquad \dot{f}^H(a) = b'$$

Technically: $\langle a, b\rangle \in \dot{f}^H \qquad \langle a, b'\rangle \in \dot{f}^H$

But recall $p \Vdash$ "$\dot{f}$ is a function", and $p \in H$ because $r \leq q, q' \leq p$. Contradiction. $\square$ (1).

Now given $a \in A$ let

$$D_a = \{ q \leq p \mid (\exists b \in B)(q \Vdash \dot{f}(\check{a}) = \check{b}) \}$$

<u>Exercise</u> D_a is an open subset of $p{\downarrow}$.