MATH 281C SPRING 2020 L10

Recall we have a κ-c.c. poset $\mathbb{P} \in M$, sets $A, B \in M$, a $\mathbb{P}$-term $\dot{f}$ and a condition $p \in \mathbb{P}$ s.t. $p \Vdash \dot{f} : \check{A} \to \check{B}$.
We defined for $a \in A$:

$$D_a = \{ q \leq p \mid (\exists b \in B)\ q \Vdash \dot{f}(\check{a}) = \check{b} \}$$

Then D_a is an open set in $\mathbb{P}$. (Easy from the properties of $\Vdash$.) Now using AC pick $X_a \subseteq D_a$ some maximal antichain in D_a. Now the point is:

(2) if $b \in B$ and $q \leq p$ is s.t. $q \Vdash \dot{f}(\check{a}) = \check{b}$ then there is some $q' \in X_a$ s.t. $q' \Vdash \dot{f}(\check{a}) = \check{b}$

Proof of (2): Notice $q \in D_a$. Because X_a is a MAC in D_a there is some $q' \in X_a$ s.t. $q' \parallel q$. This means: there is some $b' \in B$ s.t. $q' \Vdash \dot{f}(\check{a}) = \check{b}'$. Now pick any $r \leq q, q'$. Then $r \Vdash (\dot{f}(\check{a}) = \check{b} \wedge \dot{f}(\check{a}) = \check{b}')$. So $r \Vdash \check{b} = \check{b}'$. Hence $b = b'$.
(Technically: $r \Vdash (\langle \check{a}, \check{b} \rangle \in \dot{f} \wedge \langle \check{a}, \check{b}' \rangle \in \dot{f})$
But $r \Vdash \dot{f}$ is a function, so
$r \Vdash (\langle \check{a}, \check{b} \rangle \in \dot{f} \wedge \langle \check{a}, \check{b}' \rangle \in \dot{f}) \to \check{b} = \check{b}'$)

$\square$ (2)

(2) Tells us that if $G \ni p$ is $(\mathbb{P}, M)$-generic and $b = \dot{f}^G(a)$ then $q' \Vdash \dot{f}(\check{a}) = \check{b}$ for some $q' \in X_a$. This is because by (B) in the Forcing Thm we know that there is some $q_0 \in G$ s.t. $q_0 \Vdash \dot{f}(\check{a}) = b$. But because $p \in G$ we can find $q \leq p, q_0$ s.t. $q \Vdash \dot{f}(\check{a}) = b$. Then by (2) $q' \Vdash \dot{f}(\check{a}) = \check{b}$ some $q' \in X_a$.

In other words: the values of $\dot{f}$ forced by elements of

X_a are all possible values of $\dot{f}^G$ where $G \in W$ ranges over all $(\mathbb{P}, M)$-generic filters. Now we can define function F:

$$F(a) = \{ b \in B \mid (\exists q \in X_a) \; q \Vdash \dot{f}(\check{a}) = \check{b} \}$$

Then $F \in M$, $F : A \to \mathcal{P}(B)$ and, by what was said immediately below the proof of (2):

$$\dot{f}^G(a) \in F(a)$$

Now define a map $g : F(a) \longrightarrow X_a$ by

$$g(b) = \text{some } q \in X_a \text{ s.t. } q \Vdash \dot{f}(\check{a}) = \check{b}$$

By the definition of $F(a)$, g is defined for all $b \in F(a)$.

Notice: $b \neq b' \implies g(b) \neq g(b')$

because the same condition cannot decide two distinct values for $\dot{f}$. Hence g is injective.

Now by the κ-c.c. of $\mathbb{P}$: In M we have $\text{card}(X_a) < \kappa$, hence $\underline{\text{card}(F(a)) < \kappa}$.

Remark We actually did not use (1). $\square$ L 11.18.

11.19. Proposition Assume $M \subseteq M'$ be transitive models of ZFC s.t. $On \cap M = On \cap M'$. Let κ be regular in M. Assume further that for every $A, B \in M$ and every $f : A \to B$ s.t. $f \in M'$ there is $F \in M$ s.t.

- $F : A \to \mathcal{P}(B)$
- $f(a) \in F(a)$ all $a \in A$
- $\text{card}^M(F(a)) < \kappa$ all $a \in A$

Then for every ordinal $\lambda \geq \kappa$:

(i) λ is regular in M' iff λ is regular in M

(ii) If $\text{cf}^M(\lambda) \geq \kappa$ then $\text{cf}^{M'}(\lambda) = \text{cf}^M(\lambda)$

(iii) λ is a cardinal in M' iff λ is a cardinal in M

Proof Exercise.

11.20. Proposition Assume $M \models ZFC$, κ is regular in M and $\mathbb{P}$ is a κ-c.c. poset in M. Then the conclusions (i)–(iii) in P 11.19. hold for M and $M' = M[G]$ when G is any $(\mathbb{P}, M)$-generic filter.

In particular: if $\mathbb{P}$ is c.c.c. then $M, M[G]$ agree on all cardinals and cofinalities.

Pf. Immediate consequence of P 11.19. $\square$ P 11.20

Now if we go back to the poset $\mathbb{P} = Add(\omega, \lambda)$ when λ is a cardinal in M. Then λ remains a cardinal in $M[G]$ and $M[G] \models 2^\omega \geq \lambda$. In fact if $\lambda = \aleph_\alpha^M$ then $\lambda = \aleph_\alpha^{M[G]}$ and $M[G] \models 2^\omega \geq \aleph_\alpha$. We checked in Fall that $Add(\omega, \lambda)$ is c.c.c.

If $\kappa \geq$ is regular then the conditions in $Add(\kappa, \lambda)$ are of size $< \kappa$. By a Δ-system argument, then $Add(\kappa, \lambda)$ is $(2^{<\kappa})^+$-c.c. Now if $M \models ZFC$ and $2^{<\kappa} = \kappa$ then $Add(\kappa, \lambda)$ is κ^+-cc. This happens for instance if $M \models GCH$, like $M = L$. In this case forcing with $Add(\kappa, \lambda)$ preserves all cardinals and cofinalities $\geq \kappa^+$.

11.21. Definition A poset $\mathbb{P}$ is κ-closed (where κ is a cardinal) iff every descending chain in $\mathbb{P}$ of length $< \kappa$ has a lower bound in $\mathbb{P}$. That is: if $\gamma < \kappa$ and $(p_\xi \mid \xi < \gamma)$ is s.t. $p_{\xi'} \leq p_\xi$ whenever $\xi < \xi'$ then there is some $p \in \mathbb{P}$ s.t. $p \leq p_\xi$ all $\xi < \gamma$.

11.22. Prop $Add(\kappa, \lambda)$ is κ-closed whenever κ is regular.

Proof If $(p_\xi \mid \xi < \gamma)$ is a descending chain in $\mathbb{P}$ then $p = \bigcup_{\xi < \gamma} p_\xi$ is a lower bound. Point: $card(p) < \kappa$, which follows from regularity of κ. $\square$ P 11.22.

11.23. Proposition Assume $M \models ZFC$ and $\mathbb{P}$ is a κ-closed poset in M; κ regular in M. Assume G is $(\mathbb{P}, M)$-generic, $\gamma < \kappa$ and $f \in M[G]$ be a function with $\mathrm{dom}(f) = \gamma$ and $\mathrm{rng}(f) \subseteq M$. Then $f \in M$.

Proof Since $f \in M[G]$ we have a $\mathbb{P}$-term $\dot{f}$ s.t. $f = \dot{f}^G$.
Assume $f \notin M$. By (B) in Forcing Thm we have a condition $p \in G$ s.t.

(1) $\quad p \Vdash \dot{f} \notin \check{M}$ (Think how to make this technically correct.)
(Meaning: All evaluations $\dot{f}^G$ are in M)

and

(2) $\quad p \Vdash \dot{f}$ is a function with domain $\check{\gamma}$

Now construct recursively elements $x_\xi \in M$ and a descending chain of conditions p_ξ as follows. Important: We work in M. By (2) we can find, using the Forcing Thm (or Property ⑥ in P. 11.16) $x_0 \in M$ and $p_0 \leq p$ s.t.

$$p_0 \Vdash \dot{f}(\check{0}) = \check{x}_0$$

If we already have x_ξ and p_ξ then $p_\xi \leq p$ by the I.H. So $p_\xi \Vdash$ "$\dot{f}$ is a function with domain $\check{\gamma}$". So we can find $x_{\xi+1} \in M$ and $p_{\xi+1} \leq p_\xi$ s.t.

$$p_{\xi+1} \Vdash \dot{f}(\check{\xi+1}) = \check{x}_{\xi+1}$$

If ξ is a limit: Notice $\xi < \gamma < \kappa$. Since $\mathbb{P}$ is κ-closed we can pick some $p'_\xi \in \mathbb{P}$ s.t. $p'_\xi \leq p_{\bar{\xi}}$ all $\bar{\xi} < \xi$. Then as above find $p_\xi \leq p'_\xi$ and $x_\xi \in M$ s.t.

$$p_\xi \Vdash \dot{f}(\check{\xi}) = \check{x}_\xi .$$

This completes the construction of $(p_\xi \mid \xi < \gamma)$ and $(x_\xi \mid \xi < \gamma)$. Since $\gamma < \kappa$ and $\mathbb{P}$ is κ-closed we can find a condition $p^* \in \mathbb{P}$ s.t. $p^* \leq p_\zeta$ for all $\zeta < \gamma$. This means:

(3) $\quad p^* \Vdash \dot{f}(\check{\zeta}) = \check{x}_\zeta$ for all $\zeta < \gamma$.

Now if we define $g : \gamma \to M$ by

$g(\zeta) = x_\zeta \qquad$ (i.e. $g = \langle x_\zeta \mid \zeta < \gamma \rangle$)

then $g \in M$. Important: We have

$p^* \Vdash \dot{f} = \check{g}$

Why: If $H \ni p^*$ is $(\mathbb{P}, M)$-generic then by (3) we have $\dot{f}^H(\zeta) = x_\zeta$ all $\zeta < \gamma$. So $\dot{f}^H(\zeta) = g(\zeta)$.

Since $g \in M$, we have

$p^* \Vdash \dot{f} \in \check{M}$.

But $p^* \leq p$, and $p \Vdash \dot{f} \notin M$ by (1).

Contradiction. $\square$ (P 11.23)

11.24. Remark By $p \Vdash \dot{f} \in \check{M}$ we mean: If $H \ni p$ is a $(\mathbb{P}, M)$-generic then $\dot{f}^H \in M$. The point is that this can be expressed in a syntactically correct way.

Here is an alternative way how to run this argument: Fix $p \in \mathbb{P}$ s.t. $p \Vdash$ "$\dot{f}$ is a function with domain $\check{\gamma}$" Notice: the above proof of P 11.23 shows:

$$D = \{ p^* \leq p \mid (\exists g \in M)(p^* \Vdash \dot{f} = \check{g}) \}$$

is dense below p.

This implies if G is $(\mathbb{P}, M)$-generic and $p \in G$

$$(\exists g \in M)(\dot{f}^G = g)$$