M281 SPRING 2020 L11

11.25. Proposition Assume $M \subseteq M'$ are transitive models of ZF s.t. $On \cap M = On \cap M'$. Let κ be regular in M and

$$^{<\kappa}M = {}^{<\kappa}M' \cap M$$

Then M, M' agree on cardinals and cofinalities $\leq \kappa$.

Proof Easy $\square$

11.26. Proposition Assume M is a transitive model of ZFC, κ is regular in M and $\mathbb{P} \in M$ is a κ-closed poset in the sense of M. If G is $(\mathbb{P}, M)$-generic then $M, M[G]$ agree on cardinals and cofinalities $\leq \kappa$.

Proof Immediate from P 11.23 and P. 11.25. $\square$ P 11.26.

11.27. Proposition If G is $(\mathrm{Add}(\kappa, \lambda), M)$-generic when κ, λ are cardinals in M and κ is regular in M then $M, M[G]$ agree on all cardinals and cofinalities

- $\leq \kappa$
- $\geq (\kappa^{<\kappa})^+$

In particular, if $M \models GCH$ then $(\kappa^{<\kappa})^+ = \kappa^+$ hence $M, M[G]$ agree on all cardinals and cofinalities. Moreover, $M[G] \models 2^\kappa \geq \lambda$. So if $\aleph_\alpha$ is regular and $\beta > \alpha$ in $M[G]$ we have $2^{\aleph_\alpha} \geq \aleph_\beta$.

Proof From P 11.26 and the calculation of saturation of $\mathrm{Add}(\kappa, \lambda)$ at the end of Fall quarter. $\mathrm{Add}(\kappa, \lambda)$ is $(\kappa^{<\kappa})^+$-c.c. $\square$ P.11.27.

11.28. Proposition Assume $\mathbb{P}$ is a poset and $\dot{a}$ is a $\mathbb{P}$-term. Assume further $\dot{x}$ is a $\mathbb{P}$-term and $p \in \mathbb{P}$ such that

$$p \Vdash \dot{x} \subseteq \dot{a}$$

Then there is

(a) a $\mathbb{P}$-term $\dot{y}$ s.t.

$$\dot{y} \subseteq \mathbb{P} \times \operatorname{rng}(\dot{a})$$

and

$$p \Vdash \dot{y} = \dot{x}$$

(b) if $M \models ZFC$, a $\mathbb{P}$-term $\dot{y}'$ of the form

$$\dot{y}' = \bigcup_{\dot{z} \in \operatorname{rng}(\dot{a})} (A_{\dot{z}} \times \{\dot{z}\})$$

where each $A_{\dot{z}}$ is an antichain in $\mathbb{P}$ s.t.

$$p \Vdash \dot{y}' = \dot{x}$$

Terms as in (a), (b) are called canonical or nice terms for subsets of $\dot{a}$.

Proof (a) Assume $\dot{x}, \dot{a}$ are $\mathbb{P}$-terms and $p \in \mathbb{P}$ s.t.

$$p \Vdash \dot{x} \subseteq \dot{a}$$

Assume $z \in \dot{x}^G$ when $p \in G$ and G is $(\mathbb{P}, M)$-generic. Because $\dot{x}^G \subseteq \dot{a}^G$: $z \in \dot{a}^G$ so we have some $(q, \dot{z}) \in \dot{a}$ s.t. $q \in G$ and $\dot{z}^G = z$. Since $z \in \dot{x}^G$ we have $\dot{z}^G \in \dot{x}^G$. By (B) of the Forcing Thm there is some $\underbrace{r \in G \text{ s.t. } r \Vdash \dot{z} \in \dot{x}}$.

Now let

$$\dot{y} = \{ (r, \dot{z}) \in \mathbb{P} \times \operatorname{rng}(\dot{a}) \mid r \Vdash \dot{z} \in \dot{x} \}$$

We have $(r, \dot{z}) \in \dot{y}$. Because $r \in G$ also $\dot{z}^G \in \dot{y}^G$.

So we proved:

$$z \in \dot{x}^G \Rightarrow z \in \dot{y}^G$$

The converse

$$z \in \dot{y}^G \Rightarrow z \in \dot{x}^G$$

is kind of trivial, using (A) of Forcing Thm.

So we have:

$$z \in \dot{x}^G \iff z \in \dot{y}^G$$

for all $(\mathbb{P}, M)$-generics G s.t. $p \in G$. This tells us

$$p \Vdash \dot{x} = \dot{y} \qquad \square \text{(a)}$$

(b) Let $\dot{y}$ be as in (a). The set

$$D_{\dot{z}} = \{ r \in \mathbb{P} \mid r \Vdash \dot{z} \in \dot{x} \}$$

is open for any $\dot{z} \in \operatorname{rng}(\dot{a})$, by basic properties of $\Vdash$. For each $\dot{z} \in \operatorname{rng}(\dot{a})$ let $A_{\dot{z}}$ be a maximal antichain in $D_{\dot{z}}$. Let $\dot{y}'$ be as in the statement of (b). Then $\dot{y}'^G = \dot{y}^G$ whenever $p \in G$, because if $G \cap D_{\dot{z}} \neq \emptyset$ then $G \cap A_{\dot{z}} \neq \emptyset$ (HW exercise). So $\dot{y}'^G = \dot{x}^G$ for all $(\mathbb{P}, M)$-generics G s.t. $p \in G$. i.e. $p \Vdash \dot{y}' = \dot{x}$. $\square$ (P 11.28.)

<u>11.29. Proposition</u> (ZFC in the ground model) Assume κ is regular, $\lambda > \kappa$ is a cardinal. Then the generic extension by $\mathrm{Add}(\kappa, \lambda)$ there is

$$\leq \left([\lambda^{<\kappa}]^{\kappa^{<\kappa}} \right)^{\kappa}$$

many subsets of κ.

<u>Proof</u> Each subset of κ is an evaluation of a name of the form $\bigcup_{\xi < \kappa} (A_\xi \times \{\check{\xi}\})$ where A_ξ is an antichain in $\mathrm{Add}(\kappa, \lambda)$. So essentially such a name is a function $f : \kappa \longrightarrow$ Antichains in $\mathrm{Add}(\kappa, \lambda)$. Each A_ξ is of size $\leq \kappa^{<\kappa}$ and $\mathrm{Add}(\kappa, \lambda) \subseteq \lambda^{<\kappa}$. So there are $\leq \left(\left[[\lambda]^{<\kappa} \right]^{\kappa^{<\kappa}} \right)^{\kappa}$ $\square$ P 11.29.

In particular, under GCH: $\kappa^{<\kappa} = \kappa$, so this number is $([\lambda]^{\kappa})^{\kappa}$.

if $cf(\lambda) > \kappa$ then $(\lambda^{<\kappa})^\kappa = \lambda$

if $cf(\lambda) \leq \kappa$ then $(\lambda^{<\kappa})^\kappa \leq (\lambda^\lambda)^\kappa = \lambda^\lambda = \lambda^+$

In the generic extension by $Add(\kappa, \lambda)$ we have $\geq \lambda$ many subsets of κ. So by the above:

if $M \models cf(\lambda) > \kappa$ then $M[G] \models 2^\kappa = \lambda$

if $M \models cf(\lambda) \leq \kappa$ then $M[G] \models \lambda \leq 2^\kappa \leq \lambda^+$

i.e. $2^\kappa \in \{\lambda, \lambda^+\}$

<u>Recall</u>: Since $M \models GCH$, the Models $M, M[G]$ agree on all cardinals and cofinalities. In particular: $M[G] \models cf(\lambda) \leq \kappa$. By König's Lemma: Necessarily $2^\kappa = \lambda^+$.

<u>Summary</u>: If κ is regular then 2^κ can be made anything (using $Add(\kappa, \lambda)$) which does not violate König's Lemma.)

Now look at the poset $Coll(\kappa, \lambda)$ when κ is regular and $\lambda \geq \kappa$. This poset is

$$(\lambda^{<\kappa})^+ \text{ - c.c.}$$

and κ-closed. So $M[G]$ agrees with M on all cardinals and cofinalities $\leq \kappa$ and $\geq (\lambda^{<\kappa})^+$. It collapses everything in between.

Special case: $\kappa = \omega$. Then $(\lambda^{<\kappa})^+ = \lambda^+$. $Coll(\omega, \lambda)$ collapses all cardinals in the interval $(\kappa, \lambda]$ and no others.

Finally look at $Coll(\kappa, <\lambda)$ when κ regular and $\lambda \geq \kappa$. This poset is in general

$$(\lambda^{<\kappa})^+ \text{- c.c.}$$

and κ-closed, but in some special cases is better:

- If $\kappa = \omega$ and λ is regular then $\mathrm{Coll}(\omega, \lambda)$ is λ-c.c. So it collapses all cardinals $< \lambda$ and no others, so
$$M[G] \models \lambda = \omega_1$$
- If κ is regular and $\lambda > \kappa$ is inaccessible then $\mathrm{Coll}(\kappa, \lambda)$ is λ-c.c., so again
$$M[G] \models \lambda = \omega_1.$$

<u>11.30. Remark</u> When we considered $\mathrm{Add}(\kappa, \lambda)$ we always took κ regular. What if κ is singular?
To think about: What happens if we force with
$$\mathrm{Add}(\aleph_\omega, 1)$$

<u>Proof of T 11.14,</u> the Generic extension Thm.
We have $M \models ZF$, $\mathbb{P} \in M$ a poset and $G \subseteq \mathbb{P}$ a $(\mathbb{P}, M)$-generic filter. WTS: $M[G] \models ZF$.

(1) We proved that $M[G]$ is transitive. This gives:
$M[G] \models$ Extensionality + Foundation.
$M[G] \models$ Existence as $M \subseteq M[G]$

(2) <u>HW Exercise</u> If $\dot{x}, \dot{y}$ are $\mathbb{P}$-terms then there are $\mathbb{P}$-terms $\dot{a}, \dot{b}$ s.t.
$$\dot{a}^G = \{\dot{x}^G, \dot{y}^G\}$$
$$\dot{b}^G = \bigcup \dot{x}^G$$
whenever G is $(\mathbb{P}, M)$-generic. This gives the Pairing + Union axioms

(3) We proved $M \subseteq M[G]$, so in particular $\omega \in M[G]$. This gives Axiom of Infinity.

It remains to prove: $M[G]$ is a model of Separation, Collection and Power Set.