1.25. **Proposition** Assume $M, M'$ are transitive models of $ZF$ s.t. $\text{On}^M = \text{On}^M'$. Let $\kappa$ be regular in $M$ and $\kappa^{\text{M}} = \kappa^{\text{M}'}$. Then $M, M'$ agree on cardinals and cofinalities $\leq \kappa$.

**Proof** Easy. \qed

1.26. **Proposition** Assume $M$ is a transitive model of $ZFC$, $\kappa$ is regular in $M$ and $\text{ReM}M$ is a $\kappa$-closed point in the sense of $M$. If $G \in (\text{P}(\kappa), M)$ - generic then $M, M[G]$ agree on cardinals and cofinalities $\leq \kappa$.

**Proof** Immediate from P 1.23 and P 1.25. \qed P 1.26.

1.27. **Proposition** If $G \in (\text{Add}(\kappa, \lambda), M)$ - generic where $\kappa, \lambda$ are cardinals in $M$ and $\kappa$ is regular in $M$ then $M, M[G]$ agree on all cardinals and cofinalities

- $\kappa \leq \kappa$
- $\kappa \geq (\kappa^{\text{c, } \kappa})^+$

In particular, if $M \models \text{GCH}$ then $(\kappa^{\text{c, } \kappa})^+ = \kappa^+$ hence $M, M[G]$ agree on all cardinals and cofinalities.

Moreover, $M[G] = 2^\kappa \geq \lambda$. So if $\kappa^+$ is regular and $\beta \geq \kappa$ in $M[G]$ we have $2^{\kappa^+} \geq \lambda^\beta$.

**Proof** From P 1.26 and the calculation of saturation of $\text{Add}(\kappa, \lambda)$ at the end of Fall quarter. $\text{Add}(\kappa, \lambda)$ is $(\kappa^{\text{c, } \kappa})^+$ - c.c. \qed P 1.27.

1.28. **Proposition** Assume $\mathbb{P}$ is a forcing and $\dot{a}$ is a $\mathbb{P}$-term.

**Assume** further $\dot{a}$ is a $\mathbb{P}$-term and $p \in \mathbb{P}$ such that $\text{p} \upharpoonright \dot{x} \leq \dot{a}$

Then there is
(a) a $P$-term $y$ s.t.
\[ y \in \prod x \text{ rng}(a) \]
and
\[ p \vdash y = x \]

(b) If $M \models ZFC$, a $P$-term $y'$ of the form
\[ y' = \bigcup_{z \in \text{rng}(a)} (A_z \times \{z\}) \]
when each $A_z$ is an antichain in $P$ s.t.
\[ p \vdash y' = x \]

Terms as in (a),(b) are called canonical or well-explicit,
for subsets of $a$.

Proof (a) Assume $x, a$ are $P$-terms and $p \in \mathcal{P}$ s.t.
\[ p \vdash x \in a \]
Assume $z \in x$ when $p \in G$ and $G \models (P, M)$-true.
Because $x \in a$ : $z \in a$ so we have some $(q, z) \in a$ s.t.
$q \in G$ and $z \in z$. Since $z \in x$ we have $z \in x$. By (B) of
the forcing Then there is some $y \in G$ s.t. $p \vdash z \in x$.

Now let
\[ y = \{ (x, z) \in (P \times \text{rng}(a)) \mid p \vdash z \in x \} \]

We have $(x, z) \in y$. Because $x \in G$ also $z \in G$.

So we proved:
\[ z \in G \Rightarrow z \in y \]

The converse
\[ z \in y \Rightarrow z \in G \]

is kind of trivial, using (A) of Forcing Then.
So we have:
\[ z \in x \iff z \in y \]

For all \((R, \mu)\) - generics \(G\) s.t. \(p \in G\). This tells us \(p \forces \text{ } x = y \). \(\square (a)\)

(b) Let \(y\) be as in (a). The set
\[ D_y = \{ z \in R \mid \forall x \in A \exists z \in x \} \]

is open for every \(z \in mg(\delta)\), by basic properties of \(H\). For each \(z \in mg(\delta)\) let \(A_z\) be a maximal antichain in \(D_y\). Let \(y'\) be as in the statement of (b). Then \(y' \in G\) whenever \(p \in G\), because if \(G \cap D_y \neq \emptyset\) then \(G \cap A_z \neq \emptyset\) (by definition). So \(y' \in G\) for all \((R, \mu)\) - generics \(G\) s.t. \(p \in G\). \(\square (\text{P11.28})\)

11.29. Proposition. Assume \(\kappa\) is regular, \(\kappa > \kappa^+\) is a cardinal.

Then the greatest cardinality by \(\text{Add}(\omega, 1)\) there is
\[ \leq (\kappa^+)^{\kappa^+}\]

may subsets of \(\kappa\).

Proof. Each subset of \(\kappa\) is an antichain of a name of the form \(\bigcup (A_y \times \mathfrak{C})\) when \(A_y\) is an antichain in \(\text{Add}(\omega, 1)\). So essentially each a name is a function \(f : \kappa \to \text{Antichains in } \text{Add}(\omega, 1)\).

Each \(A_y\) is of size \(\leq \kappa^{<\kappa}\) and \(\text{Add}(\omega, 1) \leq \kappa^{<\kappa}\).

So there are \(\leq (\kappa^{<\kappa})^{\kappa^+}\) names. \(\square\) P11.29.

In particular, under \(\text{GCH}\): \(\kappa^{<\kappa} = \kappa\), so this number is \((\kappa^+)^{\kappa^+}\).
If $\text{cf}(\lambda) > n$ then $\alpha^{<\lambda} = \lambda$

If $\text{cf}(\lambda) \leq n$ then $\alpha^{<\lambda} = \alpha = \lambda$

In the generic extension by $\text{Add}(\alpha, \lambda)$ we have

$\sum X$ may be $X$ of $\alpha$. So by the above:

If $\mathcal{M} \models \text{cf}(\lambda) > n$ then $\mathcal{M}[\mathcal{G}] \models 2^\alpha = \lambda$

If $\mathcal{M} \models \text{cf}(\lambda) \leq n$ then $\mathcal{M}[\mathcal{G}] \models X \leq 2^\alpha \leq \lambda^+$

(i.e. $2^\alpha \in \{\lambda, \lambda^+\}$)

Recall: Since $\mathcal{G}$ is CH, the Models $\mathcal{M}, \mathcal{M}[\mathcal{G}]$

agree on all cardinals and cofinalities. In particular: $\mathcal{M}[\mathcal{G}] \models \text{cf}(\lambda) = n$. By König's lemma:

$\alpha^+ = \lambda^+$

Summary: If $\alpha$ is regular then $2^\alpha$ can be

made anything (e.g. $\text{Add}(\alpha, \lambda)$ which does not

violate König's lemma.)

Now look at the $\text{coll}(\alpha, \lambda)$ when $\alpha$ is regular

and $\lambda \geq \alpha$. The point end

$(\alpha^{<\lambda})^+ \subseteq \text{c.c.}$

and $\alpha$-closed. So $\mathcal{M}[\mathcal{G}]$ agrees with $\mathcal{M}$ on all

cardinals and cofinalities $\leq \alpha$ and $\geq (\alpha^{<\lambda})^+$.

It collapses everything in between.

Special case: $\alpha = \omega$. Then $(\omega^{<\omega})^+ = \omega^+$. $\text{coll}(\omega, \lambda)$

collapses all cardinals in the interval $(\omega, \lambda]$ and

no others.

Finally look at $\text{coll}(\alpha, \alpha)$ when $\alpha$ is regular and $\lambda \geq \alpha$.

This point end in general

$(\alpha^{<\alpha})^+ \subseteq \text{c.c.}$

and $\alpha$-closed. But in some special cases is better:
• If \( n = \omega \) and \( \lambda \) is regular then \( \text{Coll}(\omega, \lambda) \) is \( \lambda \)-c.c. So it collapses all cardinals \( < \lambda \) and no others, so

\[
M[G] \models \lambda = \omega_1
\]

• If \( n \) is regular and \( \lambda > n \) is inaccessible then \( \text{Coll}(\omega, \lambda) \) is \( \lambda \)-c.c. so again

\[
M[G] \models \lambda = \omega_1
\]

1.30. Remark When we considered \( \text{Add}(\omega, 1) \) we always took \( \omega \) regular. What if \( n \) is singular?
To think about: What happens if we force with

\[
\text{Add}(\omega, \omega, 1)
\]

Proof of Th. 1.4, the General extension Theorem.
We have \( M \models ZF \), \( P \subseteq M \) a partial and \( G \subseteq P \)
a \((P, M)\)-generic filter. We show: \( M[G] \models ZF \).

1. We proved that \( M[G] \) is transitive. This gives:

\[
M[G] \models \text{Extensionality } + \text{Foundation}.
\]

\[
M[G] \models \text{Extension as } M \subseteq M[G]
\]

2. By Exercise 4 (i) if \( P \)-names then there are \( P \)-terms \( a_i, b_i \) s.t.

\[
a_i^G = \{ x^G \mid \exists \beta \in \delta. y^G \}
\]

\[
b_i^G = \bigcup \delta
\]

where \( \delta = (P, M) \)-generic. This gives

the Pairing + Union axioms
We found $\mathcal{M}\subseteq\mathcal{M}(\mathcal{C})$, so in particular we $\mathcal{M}(\mathcal{C})$. This gives Axiom of Infinity.

It remains to prove: $\mathcal{M}(\mathcal{C})$ is a model of Separation, Collection and Power Set.