Recall: We have $M \models \exists! F, \exists \Phi \in M$ a poset and $a \in (B, M)$ — generic filter $G$.

(4) $M[G] \models \text{Separation}.$

Proof: Assume on $M[G]$ we have $a, \bar{x}, \ldots, x_e$ and want to prove the existence of a set

(1) $b = \{ z \in a \mid \Phi(z, \bar{x}, \ldots, x_e) \}$

where $\Phi$ is an LST-formula.

Fix $L_\Pi$-terms $\bar{a}, \bar{x}, \ldots, \bar{x}_e$, s.t. $a = a^G$ and $x_i^G = \bar{x}_i,
(1 \leq i \leq e)$. First examine what must hold of (1) $\models$ true and $\models b$.

We must have some $(p, \bar{z}) \in \bar{a}$ s.t. $z = z^G$ and $p \in G$, and

(2) $M[G] \models \bar{z} \in a^G \land \Phi(z, \bar{x}, \ldots, x_e^G)$

So by (3) of the Facting Theorem there is some $q \in G$ s.t.

(3) $q \models \bar{z} \in a \land \Phi(z, \bar{x}, \ldots, x_e)$

So if we let

$b' = \{ (q, \bar{z}) \in IP \times \text{Alg}(a) \mid q \models \bar{z} \in a \land \Phi(z, \bar{x}, \ldots, x_e) \}$

Here we are using Separation on $M[G]$ then $\bar{z} \in b' \in b^G$ whenever $q \in G$. So using (3) we have: “$b \models b^G$.

Technically correct

(2) $\Rightarrow$ $\bar{z} \in b^G$

The converse

$\bar{z} \in b^G \Rightarrow (2)$

is easy using (4) of Facting Theorem. This shows:

$M[G] \models b^G = \{ z \in a^G \mid \Phi(z, \bar{x}, \ldots, x_e^G) \}$

$\square$ (Separation)
(5) **Collect in** Assume \( \phi(u,v,x_1 \ldots x_n) \) is an LST-form, and \( a, x_1 \ldots x_n \in M[\Gamma] \). Assume

\[ M[\Gamma] \models (\forall u \in a)(\exists v)(\phi(u,v,x_1 \ldots x_n)) \]

We want to find a bound on \( v \). Fix \( P \)-terms \( a', x_1 \ldots x_n \) s.t. \( a = a' \) and \( x_i^G = x_i \) \( i = 1 \ldots n \). If \( v \in a \), then we have some \( (p,u) \in a \) s.t. \( p \in G \) and \( u = u^G \).

Now assume \( M[\Gamma] \models (\exists w)(\phi(l', v, x_1 \ldots x_n)) \)

again by (B) in \( M \). Forcing \( \Gamma \) we have some \( q \in G \) and a \( P \)-term \( v' \) s.t.

(a) \( q \vdash \phi(l', v', x_1 \ldots x_n) \)

Recall \( v' \) ranges over \( \text{rg}(a) \), but there is no bound on \( v \). Let \( (w, u) \in M \).

\[ A = \{(q,u) \in \Gamma \times \text{rg}(a) \mid \text{There is a } P \text{-term } v' \text{ such that } q \vdash \phi(l', v', x_1 \ldots x_n) \} \]

So in \( M \) we have:

\[ \{ (q,u) \in A \mid (\exists w)(\phi(l', v', x_1 \ldots x_n)) \} \]

By Collecting in \( M \) there is some \( B \in M \) s.t.

\[ \{ (q,u) \in A \mid (\exists w)(\phi(l', v', x_1 \ldots x_n)) \} \]

Now if we let

\[ \hat{a} = \{ (q,u) \mid (\exists u')(\phi(q,u', x_1 \ldots x_n)) \} \]

then

\[ M[\Gamma] \models (\forall u \in a')(\exists v)((\forall e \in G)\phi(u,v,x_1 \ldots x_n)) \]

(Exercise) \( \square (\text{Collect in}) \)
(6) Power Set axiom 
Given a $P$-term $a$: 

we want to find a $P$-term $b$ s.t.

(1) $M(b) = \{ y \in b \mid y = \exists f \in M(\bar{a}) \}$

Here we use $\text{P in 28}$ which says that if $y \in M(\bar{a})$ and $y \subseteq \bar{a}$ then $y = \exists f \in M(\bar{a})$. 

This says we only need to consider $P$-terms in $\text{P}(\text{P} \times \text{M}(\bar{a}))$; here we use the Power Set Axiom in $M$. If we just let 

$b' = \{ (x, y) \mid y \in M(\text{P} \times \text{M}(\bar{a})) \}$

then $b'$ is a $P$-term and $M(b') = \{ y \in b' \}$.

This suffices for the verification of the Power Set Axiom in $M(b)$. To construct $b$ as in (1): Execute:

$\Box$ (Power Set)

This verifies the following:

$M \models \exists \Rightarrow M(b) \models \exists f$

(6) Axiom of Choice 
Assume $M \models AC$.

Let $a \in M(b)$. Again $a = \bar{a}$ for some $P$-term $a$. By $AC$ in $M$, we have a bijection between some $M$-cardinal $\kappa$ and $a$, so say the sequence 

$(\kappa, \bar{x}) \in M$

is such a bijection. Recall $\bar{a} = \{ (\kappa, \bar{x}) \mid x \in \bar{a} \}$.

So in $M(b)$ we can define a partial function $f: \kappa \rightarrow a$.
\[ f(3) = \overline{\omega}_3 \quad \text{for} \quad 3 \quad \text{s.t.} \quad 3 \in G \]

i.e. \( \text{dom}(f) = \{3 \in G | 3 \in G \} \). Notice \( f \)

is a surjection onto \( \bar{\omega} \). Since \( \text{dom}(f) \) is a set

of ordinals, \( \bar{\omega} \) can be well-ordered. \( \Box \) (M. C)

This proves:

\[ \text{M}^\text{2FC} \implies \text{M}(\lambda) \models \text{ZFC} \]

\( \Box \) (Theorem 11.14)

11.31. Proposition (Maximality Principle) Assume

\( \varphi(u, v_1, \ldots, v_e) \) is an LST-formula, \( P \in \text{M} \) is a poset,

\( p \in P \) and \( x_1, \ldots, x_e \) are \( P \)-terms.

(a) Assume \( \text{M} \models \text{ZFC} \). If

\[ p \models \exists u \varphi(u, v, x_1, \ldots, x_e) \]

then there is a \( P \)-term \( \bar{y} \) s.t.

\[ p \models \varphi(\bar{y}, x_1, \ldots, x_e) \]

(b) Assume \( \text{M} \models \text{ZF} \). If

\[ p \models \exists u \varphi(u, v, x_1, \ldots, x_e) \]

then there is a \( P \)-term \( \bar{y} \) s.t.

\[ p \models \varphi(\bar{y}, x_1, \ldots, x_e) \]

Proof (a) By P. 11.16. The set

\[ D = \{ q \in P | q \models \varphi(\bar{y}', x_1, \ldots, x_e) \text{ for some } \bar{P}\text{-term } \bar{y}' \} \]

is open dense below \( p \).
Pick \( A \subseteq D \) some maximal antichain in \( D \), and for each each \( q \in A \) pick some \( P\)-term \( y_q \) s.t.:

\[ q \vdash \exists \gamma (y_q, x_1 \ldots x_e) \]

Here we use AC. Now construct a \( P\)-term \( y \) s.t.:

\[ q \vdash y = y_q \quad \text{whereas } q \in A \]

(Exner) Then

\[ q \vdash \varphi (y, x_1 \ldots x_e) \]

Because \( \varphi \) is a \( (P,M) \)-sentence with \( p \in M \)

then \( \varphi \) contains exactly one \( q \in A \). Then \( y^G = \hat{y}^G \)

so \( M(G) = \varphi (y^G, x_1^G \ldots x_e^G) \). \( \Box (a) \)

(b) Now assume

\[ p \vdash \exists ! \gamma (y, x_1 \ldots x_e) \]

Assume \( G \ni p \) is \( (P,M) \)-generic and \( y \in M(G) \)

s.t.:

\[ M(G) = \varphi (y^G, x_1^G \ldots x_e^G) \]

What does \( \exists ! \) mean that \( z \in q \)? We have a \( P\)-term \( y \)' s.t. \( y = y^G \) and also a \( P\)-term \( z \in M(G)(y) \) s.t. \( z^G = z \). So

\[ M(G) = z^G \in y^G \land \varphi (y^G, x_1^G \ldots x_e^G) \]

By (b) in Generic Then we have some \( q \in G \) s.t.

(1) \[ q \vdash z \in y \land \varphi (y', x_1 \ldots x_e) \]

So we can define our \( P\)-term \( y \) as follows
(2) \((q, \hat{z}) \in \gamma \quad \Rightarrow \quad \text{there is a } P\text{-term } \tilde{y} \quad \text{s.t.} \quad q \vdash \exists \tilde{y} \cdot \varphi(\tilde{y}, \tilde{x}_1, \ldots, \tilde{x}_k)\)

This makes sense for the following reasons:

If \(q \vdash \varphi(y', \tilde{x}_1, \ldots, \tilde{x}_k)\) and \(\forall \tilde{x} \cdot \varphi(y'', \tilde{x}_1, \ldots, \tilde{x}_k)\)

and \(q \in G^3\), then

\[ M[G] \models \varphi(y', \tilde{x}_1, \ldots, \tilde{x}_k) \wedge \varphi(y'', \tilde{x}_1, \ldots, \tilde{x}_k) \]

Since \(p \vdash (\exists \tilde{x}) \varphi(y, \tilde{x}_1, \ldots, \tilde{x}_k)\) we have

\(y'' = y'\). This means: Any two \(y', y''\) which satisfy the forcing relation on the right side in (2) are consistent on \(\exists \tilde{x} \cdot \varphi(\tilde{y}, \tilde{x})\).

More precisely, the way we defined \(\gamma\) states:

- if \(y'\) is a \(P\)-term as in (2) then \(y'^G \subseteq y^G\).

The converse \(y^G \subseteq y'^G\) follows by (A) of Forcing Theorem from (2). (Easy half.) \(\square (6)\)

\[ \square (P \text{ II. 37.}) \]