M281 SPRING 2020 L12

Recall: We have $M \models ZF$, $\mathbb{P} \in M$ a poset and a $(\mathbb{P}, M)$-generic filter G.

(4) $M[G] \models$ Separation.

Proof Assume in $M[G]$ we have $a, x_1 \dots x_e$ and want to prove the existence of a set

(1) $\qquad b = \{ z \in a \mid \varphi(z, x_1 \dots x_e) \}$

when φ is an LST-formula.

Fix $\mathbb{P}$-terms $\dot{a}, \dot{x}_1, \dots \dot{x}_e$ s.t. $a = \dot{a}^G$ and $x_i = \dot{x}_i^G$ $i = 1, \dots e$. First examine what must hold if (1) is true and $z \in b$.

We must have some $(p, \dot{z}) \in \dot{a}$ s.t. $z = \dot{z}^G$ and $p \in G$, and

(2) $M[G] \models \dot{z}^G \in \dot{a}^G \wedge \varphi(\dot{z}^G, \dot{x}_1^G, \dots, \dot{x}_e^G)$

So by (B) of the Forcing Thm there is some $q \in G$ s.t.

(3) $\quad q \Vdash \dot{z} \in \dot{a} \wedge \varphi(\dot{z}, \dot{x}_1 \dots \dot{x}_e)$

So if we let

$$\dot{b} = \{ (q, \dot{z}) \in \mathbb{P} \times \operatorname{rng}(\dot{a}) \mid q \Vdash \dot{z} \in \dot{a} \wedge \varphi(\dot{z}, \dot{x}_1 \dots \dot{x}_e) \}$$

(Here we are using Separation in M) then $\dot{z}^G \in \dot{b}^G$ whenever $q \in G$. So using (3) we have: "$b \subseteq \dot{b}^G$"

Technically correctly

$$(2) \implies \dot{z}^G \in \dot{b}^G$$

The converse

$$\dot{z}^G \in \dot{b}^G \implies (2)$$

is easy using (A) of Forcing Thm. This shows:

$$M[G] \models \dot{b}^G = \{ z \in \dot{a}^G \mid \varphi(z, \dot{x}_1 \dots \dot{x}_e) \}$$

$\square$ (Separation)

(5) <u>Collection</u> Assume $\varphi(u, v, v_1 \ldots v_e)$ is an LST-formula, and $a, x_1 \ldots x_e \in M[G]$. Assume

$$M[G] \models (\forall u \in a)(\exists v)\, \varphi(u, v, x_1 \ldots x_e)$$

We want to find a bound on v. Fix $\mathbb{P}$-terms $\dot{a}, \dot{x}_1 \ldots \dot{x}_e$ s.t. $a = \dot{a}^G$ and $\dot{x}_i^G = x_i$ $i = 1, \ldots e$. If $u \in a = \dot{a}^G$ then we have some $(p, \dot{u}) \in \dot{a}$ s.t. $p \in G$ and $u = \dot{u}^G$.

Now assume

$$M[G] \models (\exists v)\, \varphi(\dot{u}^G, v, \dot{x}_1 \ldots \dot{x}_e)$$

again by (B) in Forcing Thm we have some $q \in G$ and a $\mathbb{P}$-term $\dot{v}$ s.t.

(1) $q \Vdash \varphi(\dot{u}, \dot{v}, \dot{x}_1 \ldots \dot{x}_e)$

Recall $\dot{u}$ ranges over $\mathrm{rng}(\dot{a})$, but there is no bound on $\dot{v}$. Let (in M)

$$A = \{ (q, \dot{u}) \in \mathbb{P} \times \mathrm{rng}(\dot{a}) \mid \text{There is a } \mathbb{P}\text{-term } \dot{v} \text{ such that } q \Vdash \varphi(\dot{u}, \dot{v}, \dot{x}_1 \ldots \dot{x}_e) \}$$

So in M we have:

$$(\forall (q, \dot{u}) \in A)(\exists \dot{v})(\dot{v} \text{ is a } \mathbb{P}\text{-term and } q \Vdash \varphi(\dot{u}, \dot{v}, \dot{x}_1 \ldots \dot{x}_e))$$

By Collection in M there is some $B \in M$ s.t.

$$(\forall (q, \dot{u}) \in A)(\exists \dot{v} \in B)(\dot{v} \text{ is a } \mathbb{P}\text{-term and } q \Vdash \varphi(\dot{u}, \dot{v}, \dot{x}_1 \ldots \dot{x}_e))$$

Now if we let

$$\dot{b} = \{ (q, \dot{v}) \mid (\exists \dot{u})((q, \dot{u}) \in A) \}$$

then

$$M[G] \models (\forall u \in \dot{a}^G)(\exists v \in \dot{b}^G)\, \varphi(u, v, \dot{x}_1^G, \ldots \dot{x}_e^G)$$

(Exercise) $\square$ (Collection)

(9) <u>Power Set axiom</u> Given a $\mathbb{P}$-term $\dot a$ we want to find a $\mathbb{P}$-term $\dot b$ s.t.

$$(1) \qquad M[G] \models \dot b^G = \mathcal{P}(\dot a^G)$$

Here we use P 11.28 which says that if $y \in M[G]$ and $y \subseteq \dot a^G$ then $y = \dot y^G$ s.t. $\dot y \subseteq \mathbb{P} \times \mathrm{rng}(\dot a)$. This says we only need to consider $\mathbb{P}$-terms in $\mathcal{P}(\mathbb{P} \times \mathrm{rng}(\dot a))$; here we use the Power Set Axiom in M. If we just let

$$\dot b' = \{ (1_{\mathbb{P}}, \dot y) \mid \dot y \in \mathcal{P}(\mathbb{P} \times \mathrm{rng}(\dot a)) \}$$

then $\dot b'$ is a $\mathbb{P}$-term and $M[G] \models \mathcal{P}(\dot a^G) \subseteq \dot b'^G$.
This suffices for the verification of the Power Set Axiom in $M[G]$. To construct $\dot b$ as in (1): Exercise.

$\square$ (Power Set)

This verifies the following:

$$M \models ZF \Rightarrow M[G] \models ZF$$

(10) <u>Axiom of Choice</u> Assume $M \models AC$.
Let $a \in M[G]$. Again $a = \dot a^G$ for some $\mathbb{P}$-term $\dot a$. By AC in M, we have a bijection between some M-cardinal γ and $\dot a$, so say the sequence

$$\langle (p_\xi, \dot z_\xi) \mid \xi < \gamma \rangle \in M$$

is such a bijection. Recall $\dot a^G = \{ \dot z_\xi^G \mid p_\xi \in G \}$.
So in $M[G]$ we can define a partial function

$$f : \gamma \longrightarrow a$$

by

$$f(\xi) \simeq \dot{z}_\xi^G \quad \text{for } \xi \text{ s.t. } p_\xi \in G$$

i.e. $\mathrm{dom}(f) = \{\xi < \gamma \mid p_\xi \in G\}$. Notice f is a surjection onto $\dot{a}^G$. Since $\mathrm{dom}(f)$ is a set of ordinals, $\dot{a}^G$ can be well-ordered. $\square$ (AC)

This proves:

$$M \models ZFC \implies M[G] \models ZFC$$

$\square$ (Theorem 11.14)

<u>11.31. Proposition</u> (Maximality Principle) Assume $\varphi(u, v_1 \ldots v_e)$ is an LST-formula, $\mathbb{P} \in M$ is a poset, $p \in \mathbb{P}$ and $\dot{x}_1 \ldots \dot{x}_e$ are $\mathbb{P}$-terms.

(a) Assume $M \models ZFC$. If

$$p \Vdash (\exists v)\, \varphi(v, \dot{x}_1 \ldots \dot{x}_e)$$

then there is a $\mathbb{P}$-term $\dot{y}$ s.t.

$$p \Vdash \varphi(\dot{y}, \dot{x}_1 \ldots \dot{x}_e)$$

(b) Assume $M \models ZF$. If

$$p \Vdash (\exists! v)\, \varphi(v, \dot{x}_1 \ldots \dot{x}_e)$$

then there is a $\mathbb{P}$-term $\dot{y}$ s.t.

$$p \Vdash \varphi(\dot{y}, \dot{x}_1 \ldots \dot{x}_e)$$

<u>Proof</u> (a) By P 11.16. the set

$$D = \{q \le p \mid q \Vdash \varphi(\dot{y}', \dot{x}_1 \ldots \dot{x}_e) \text{ for some } \mathbb{P}\text{-term } \dot{y}'\}$$

is open dense below p.

Pick $A \subseteq D$ some maximal antichain in D, and for each each $q \in A$ pick some $\mathbb{P}$-term $\dot{y}_q$ s.t.

$$q \Vdash \varphi(\dot{y}_q, \dot{x}_1 \dots \dot{x}_e)$$

Here we use AC. Now construct a $\mathbb{P}$-term $\dot{y}$ s.t.

$$q \Vdash \dot{y} = \dot{y}_q \qquad \text{whenever } q \in A$$

(Exercise). Then

$$p \Vdash \varphi(\dot{y}, \dot{x}_1 \dots \dot{x}_e)$$

Because if G is a $(\mathbb{P}, M)$-generic with $p \in G$ then G contains exactly one $q \in A$. Then $\dot{y}_q^G = \dot{y}^G$ so $M[G] \models \varphi(\dot{y}^G, \dot{x}_1^G, \dots \dot{x}_e^G)$. $\square$ (a)

(b) Now assume

$$p \Vdash (\exists! v) \varphi(v, \dot{x}_1, \dots \dot{x}_e)$$

Assume $G \ni p$ is $(\mathbb{P}, M)$-generic and $y \in M[G]$ is s.t.

$$M[G] \models \varphi(y, \dot{x}_1^G \dots \dot{x}_e^G)$$

What does it mean that $z \in y$? We have a $\mathbb{P}$-term $\dot{y}'$ s.t. $y = \dot{y}'^G$ and also a $\mathbb{P}$-term $\dot{z} \in \mathrm{rng}(\dot{y}')$ s.t. $\dot{z}^G = z$. So

$$M[G] \models \dot{z}^G \in \dot{y}'^G \wedge \varphi(\dot{y}'^G, \dot{x}_1^G, \dots \dot{x}_e^G)$$

By (B) in Forcing Thm we have some $q \in G$ s.t.

(1) $\quad q \Vdash \dot{z} \in \dot{y}' \wedge \varphi(\dot{y}', \dot{x}_1 \dots \dot{x}_e)$

So we can define an $\mathbb{P}$-term $\dot{y}$ as follows

(2) $(q, \dot{z}) \in \dot{y} \iff$ there is a $\mathbb{P}$-term $\dot{y}'$ s.t.

$$q \Vdash \dot{z} \in \dot{y}' \wedge \varphi(\dot{y}', \dot{x}_1 \ldots \dot{x}_\ell)$$

This makes sense for the following reasons:

If $q \Vdash \varphi(\dot{y}', \dot{x}_1 \ldots \dot{x}_\ell)$ and $r \Vdash \varphi(\dot{y}'', \dot{x}_1 \ldots \dot{x}_\ell)$ and $q, r \in G \ni p$ then

$$M[G] \models \varphi(\dot{y}'^G, \dot{x}_1^G \ldots \dot{x}_\ell^G) \wedge \varphi(\dot{y}''^G, \dot{x}_1^G \ldots \dot{x}_\ell^G)$$

Since $p \Vdash (\exists! v)\varphi(v, \dot{x}_1 \ldots \dot{x}_\ell)$ we have $\dot{y}'^G = \dot{y}''^G$. This means: Any two $\dot{y}', \dot{y}''$ which satisfy the forcing relation on the right side in (2) are consistent on "$\dot{z}^G \in \dot{y}'^G$".

<u>More precisely</u> the way we defined $\dot{y}$ states: if $\dot{y}'$ is a $\mathbb{P}$-term as in (2) then $\dot{y}'^G \subseteq \dot{y}^G$. The converse $\dot{y}^G \subseteq \dot{y}'^G$ follows by (A) of Forcing Theorem from (2). (Easy half.) $\square$ (b)

$\square$ (P 11.31.)