M281 SPRING 2020 L13

11. EXAMPLES OF FORCING

Convention: We force over V. So we have V in place of M.

① Forcing CH κ is regular. Let $\mathbb{P}$ be defined as follows

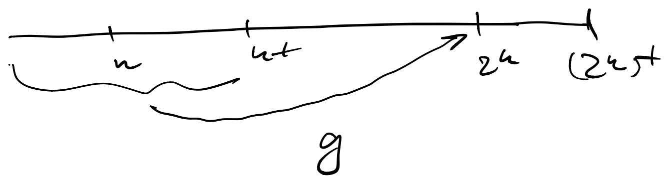
- Conditions are functions $p : a \to \mathcal{P}(\kappa)$ where $a \subseteq \kappa^+$ and $\mathrm{card}(a) \leq \kappa$.
- Ordering $p \leq q \iff p \supseteq q$

Properties of $\mathbb{P}$: (a) $\mathbb{P}$ is κ^+-closed. So in particular $\mathbb{P}$ does not add any function $f : \kappa \to V$. So if G is $(\mathbb{P}, V)$-generic then $\mathcal{P}(\kappa)^{V[G]} = \mathcal{P}(\kappa)^V$.

(b) $\mathbb{P}$ has the $(2^\kappa)^+$-Knaster property. So $\mathbb{P}$ is $(2^\kappa)^+$-c.c. hence $\mathbb{P}$ preserves all cardinals and cofinalities $> 2^\kappa$. (Exercise)

(c) Letting $g = \bigcup G$, $g : \kappa^+ \to \mathcal{P}(\kappa)$ is a surjection. This follows by a density argument. (Exercise.)

Conclusion: $V[G] \models 2^\kappa = \kappa^+$. So if $\kappa = \omega$ then $V[G] \models CH$



Notice: If $V \models 2^\kappa = \kappa^+$ then $\mathbb{P}$ does not collapse any cardinals.

(2) <u>Forcing $\diamondsuit_\kappa$</u> Assume κ is regular, $\kappa > \omega_1$.

Let $\mathbb{P}$ be defined as follows:

<u>Conditions</u> $p = (a_\xi \mid \xi < \alpha)$ where $\alpha < \kappa$ and $a_\xi \subseteq \xi$ for all $\xi < \alpha$

<u>Ordering</u> $p \leq q \iff p \supseteq q$

(a) $\mathbb{P}$ is κ-closed (Exercise, easy)

So if G is $(\mathbb{P}, V)$-generic then $V, V[G]$ agree on cardinals + cofinalities $\leq \kappa$. In particular κ is regular in $V[G]$.

Let
$$A_G = \bigcup G$$
We show:
$$A_G \text{ is a } \diamondsuit_\kappa \text{ - sequence in the sense of } V[G].$$

We will use the following

<u>12.1. Proposition</u> Assume $\mathbb{Q}$ is a κ-closed forcing where κ is regular, $\gamma < \kappa$, $p \in \mathbb{Q}$ and $\dot{x}$ is a $\mathbb{Q}$-term s.t.
$$p \Vdash \dot{x} \subseteq \check{\gamma}$$
Then there are $q \leq p$ and a set $a \subseteq \gamma$ s.t.
$$q \Vdash \dot{x} = \check{a}$$
<u>Proof</u> HW. Point using the κ-closure, we can fix the truth value of "$\check{\xi} \in \dot{x}$" by single condition for all $\xi < \gamma$.

$\square$ (P12.1.)

To prove that A_G is a $\diamondsuit_\kappa$-sequence in $V[G]$ it suffices to do the following. Given a condition $p \in \mathbb{P}$ and $\mathbb{P}$-terms $\dot{X}, \dot{C}$ s.t.

(1) $p \Vdash \dot{X} \subseteq \check{\kappa} \wedge \dot{C}$ is a club in $\check{\kappa}$

we find a condition $p^* \leq p$ and some $\gamma \in \mathrm{On}_\kappa$ s.t.

(2) $p^* \Vdash \check{\gamma} \in \dot{C} \wedge (A_G)_{\check{\gamma}} = \dot{X} \cap \check{\gamma}$

One way of thinking about this: First, using (B) in Forcing theorem, obtain (1) with $p \in G$. Then the proof of (2) shows that (2) holds for densely many $p^* \leq p$. Since $p \in G$, G must also contain one of those p^*. Then apply (A) in the Forcing Thm.

So assume (1) holds. We construct a descending chain of conditions $(p_n \mid n \in \omega)$ and an increasing sequence of ordinals $(\alpha_n \mid n < \omega)$ as follows.

Because $p \Vdash$ "$\dot{C}$ is a club in κ" we can find some $p_0' \leq p$ and $\alpha_0 > \mathrm{lh}(p)$ s.t.

$$p_0' \Vdash \alpha_0 \in \dot{C} \qquad \text{(Application of Forcing Thm, Exercise)}$$

Now using P 1.11 Find some condition $p_0'' \leq p_0'$ and some $x_0 \subseteq \alpha_0$ s.t.

$$p_0'' \Vdash \dot{X} \cap \check{\alpha}_0 = \check{x}_0$$

If $\mathrm{lh}(p_0'') < \alpha_0$ extend p_0'' to some $p_0 \leq p_0''$ s.t. $\mathrm{lh}(p_0) \geq \alpha_0$. Otherwise let $p_0 = p_0''$.

Now if x, p_n, α_n are already found, find $p_{n+1} \leq p_n$ and $\alpha_{n+1} > \alpha_n$ similarly, so that

(3) $\alpha_{n+1} \geq \mathrm{lh}(p_n)$

(4) $p_{n+1} \Vdash \dot{X} \cap \check{\alpha}_{n+1} = \check{x}_{n+1}$

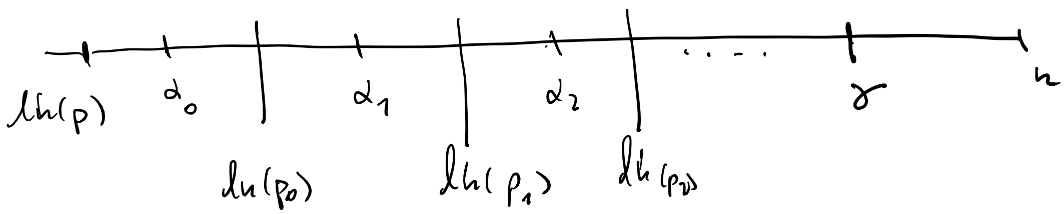
At the step ω, let

$$p_\omega = \bigcup_{n\in\omega} p_n \qquad lh(p_\omega) = \gamma$$

$$x_\omega = \bigcup_{n\in\omega} x_n \qquad x_{n+1} \text{ end-extends } x_n \text{ by } (4)$$

$$\gamma = \sup_n \alpha_n$$

$$p^* = p_\omega \cup \{\langle \gamma, x_\omega \rangle\}$$



(5) Since $p_\omega \le p_n$ for all n:

$$p_\omega \Vdash \alpha_n \in \dot{C} \qquad \text{all } n$$

Since $\gamma = \sup_n \alpha_n$ and $p_\omega \Vdash \dot{C}$ is a club in $\check{\kappa}$:

$p_\omega \Vdash \check{\gamma}$ is a limit point of $\dot{C}$

so $p_\omega \Vdash \check{\gamma} \in \dot{C}$

(6) By (4) we get

$$p_\omega \Vdash \dot{X} \cap \check{\alpha}_n = \check{x}_n \qquad \text{all } n$$

so $p_\omega \Vdash \dot{X} \cap \check{\gamma} = \check{x}_\omega$

It follows:

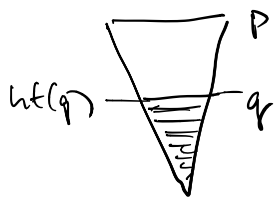
(7) $$p^* \Vdash \dot{X} \cap \check{\gamma} \in \dot{C} \wedge \dot{X} \cap \check{\gamma} = \check{p^*(\gamma)}$$

Now notice (7) says the same thing as (2), as $(A_G)_\gamma = p^*(\gamma)$. □ (2)

③ <u>Suslin Tree</u> (Jech) The poset is as follows:

<u>Conditions</u> $p \in \mathbb{P}$ iff p is a countable tree on a set ordinals below ω_1. Denote the corresponding tree ordering by $<_p$. We also require ⓐ + ⓑ below.

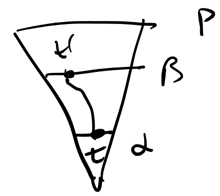
<u>Ordering</u> End-extension: $p \leq q$ iff q consists of all levels of p up to $\mathrm{ht}(q)$, and $<_q \; = \; <_p$ restricted to these levels.



<u>Addition to conditions:</u>

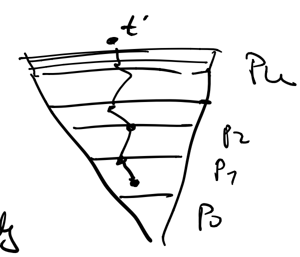
ⓐ $\mathrm{ht}(p)$ is a successor ordinal, and → every node not on the last level is a splitting node.

ⓑ For any $\alpha < \beta < \mathrm{ht}(p)$ and any $t \in \mathrm{lev}_\alpha(p)$ there is $t' \in \mathrm{lev}_\beta(p)$ s.t. $t <_p t'$.



(a) $\mathbb{P}$ is ω_1-closed (Exercise)

If $(p_n \mid n \in \omega)$ is a descending chain:
Let p_ω = "$\bigcup_n p_n$"
then obtain p' a lower bound by adding a last level using ⓐ + ⓑ. For any $t \in p_\omega$ add a node t' the last level.



ⓑ ω_1 is preserved in the generic extension, i.e. $\omega_1^{V[G]} = \omega_1^V$ whenever G is $(\mathbb{P}, V)$-generic.

ⓑ For a $(\mathbb{P}, V)$-generic G let
$$T_G = (\omega_1, \bigcup \{ <_p \mid p \in G \})$$

Then T_G is tree on ω_1. (Exercise.) There is a density argument. Show:

- For each $\alpha < \omega_1$

$$D_\alpha = \{p \in \mathbb{P} \mid \alpha \text{ is a node in } p\}$$

is dense in $\mathbb{P}$

Next show: $<_{T_G}$ is a tree ordering (Exercise)

(c) The levels of T_G are countable, as the ordering of $\mathbb{P}$ is by end-extension.

(d) $ht(T_G) = \omega_1$. (Exercise) Again a density argument. By induction on α, using ω_1-closure of $\mathbb{P}$ for limit α, show:

$$E_\alpha = \{p \in \mathbb{P} \mid ht(p) \geq \alpha\}$$

is dense in $\mathbb{P}$.

<u>Summary so far</u>:

(e) T_G is an ω_1-tree in $V[G]$.

We <u>show</u>: T_G is a Suslin tree in $V[G]$.