12. EXAMPLES OF FORCING

Convention: We face over $V$. So we have $V$ in place of $M$.

(1) Forcing $CH$ is regular. Let $P$ be defined as follows:
   - Conditions are functions $p : a \rightarrow \mathbb{P}(a)$ where $a \leq n^+$ and $\text{card}(a) \leq \aleph_1$.
   - Order $p \leq q \iff p \geq q$

Properties of $P$:
   (a) $P$ is $n^+$-closed. So in particular $P$ does not add any function $f : n \rightarrow V$.
   So if $G$ is $(P, V)$-generic then $\mathbb{P}(G)^V = \mathbb{P}(G)^n$.

   (b) $P$ has the $(2^n)^+ - $-closure property. So $P$ is $(2^n)^-$-closed.
   Hence $P$ preserves all cardinals and cofinalities $\geq 2^n$.
   (Exercise)

   (c) Let $g = U G$, $g : n^+ \rightarrow \mathbb{P}(n)$ is a surjection.
   This follows by a density argument. (Exercise.)

Conclusion: $V[G] \models 2^n = n^+$. So if $n = \omega$ then $V[G] \models CH$

\[ n \xrightarrow{\mathbb{P}} n^+ \xrightarrow{\mathbb{P}} 2^n \xrightarrow{\mathbb{P}} (2^n)^+ \]

Notice: if $V \models 2^n = n^+$ then $P$ does not collapse any cardinals.
(2) Facing $\Omega_\kappa$. Assume $\kappa$ is regular, $\kappa > \omega_1$.

We define $\mathcal{P}$ as follows:

**Conditions**  
$p = (a_\gamma | \exists \delta | a_\delta \in a_\gamma$ and $a_\delta \in \gamma$  
for all $\gamma < \delta$

**Ordering** $p \leq q \iff p \subseteq q$

(a) $\mathcal{P}$ is $\kappa$-closed (generic, easy)

So if $G \subseteq (\mathcal{P}, \cap) -$generic then $V, V[G]$ agree on

| cardinalities + cofinalities | $\leq \kappa$. In particular $\kappa$ is regular in $V[G]$.

Let $A_G = \bigcup G$

We show:

$A_G$ is a $\langle \diamondsuit \rangle$ sequence in the sense of $V[G]$.

We will use the following

12.1 Proposition. Assume $\mathcal{Q}$ is a $\kappa$-closed forcing where $\kappa$ is regular, $\gamma < \kappa$, $p \in \mathcal{Q}$ and $x$ is a $\mathcal{Q}$-term s.t.

$p \Vdash x \subseteq \gamma$

Then then are $q \leq p$ and a set $a \subseteq x$ s.t.

$q \Vdash x = a$

Proof. Taut. Point using the $\kappa$-closedness, we can fix the 

**true value of** $\exists \gamma' x$ by single conditions below $\gamma'$.

$\square$ (Prop 12.1.)

To prove that $A_G$ is a $\langle \diamondsuit \rangle$-sequence in $V[G]$ it suffices to do the following. Given a condition $p \in \mathcal{P}$ and $1^{\mathcal{P}}$-terms $x, y$ s.t.
(1) \[ \varphi \vdash \forall x \in \mathcal{U} \wedge C \text{ is a club in } \mathcal{U} \]
we find a condition \( p^* \leq \varphi \) and some \( \gamma \in \mathcal{U} \) such that
\[ \varphi_{\downarrow \gamma} \vdash \forall y \in C \wedge (A_G)\gamma = \check{x} \wedge \check{y} \]

One way of thinking about this is to fix \( C \) first and then apply (1). Then the proof of (2) shows that (2) holds for densely many \( p^* \leq \varphi \).
Since \( p \in G \), \( G \) must also contain one of these \( p^* \).
Thus apply (A) in the Forcing Theorem.

So assume (1) holds. We construct a descending chain of conditions \( \langle p_n \mid n \in \omega \rangle \) and an increasing sequence of ordinals \( \langle x_n \mid n \in \omega \rangle \) as follows.

Because \( \varphi \vdash \forall x \in \mathcal{U} \wedge C \text{ is a club in } \mathcal{U} \) we can find
some \( p_0 \leq \varphi \) and \( x_0 > \text{lh}(p_{ \downarrow p}) \) s.t.
\[ p_0 \vdash x_0 \in C \quad \text{(Application of Forcing Theorem, Exercise)} \]

Now using (A) find some
condition \( p_0'' \leq p_0 \) and some \( x_0 \leq x_0'' \) s.t.
\[ p_0'' \vdash \forall \check{x} \wedge \check{x_0} = \check{x_0} \]

If \( \text{lh}(p_0) < x_0 \) extend \( p_0'' \) to some \( p_0 \leq p_0'' \) s.t.
\[ \text{lh}(p_0) \geq x_0 \]. Otherwise let \( p_0 = p_0'' \).

Now if \( x, p_n, d_n \) are already found, find \( p_{n+1} \leq p_n \)
and \( d_{n+1} > d_n \) similarly, so that

(3) \[ d_{n+1} \geq \text{lh}(p_n) \]

(4) \[ p_{n+1} \vdash \forall \check{x} \wedge \check{d_{n+1}} = \check{x_{n+1}} \]
At the step $w$, let

$$P_w = \bigcup_{n \in w} P_n$$

$$x_w = \bigcup_{n \in w} x_n$$

$$\delta = \sup_n \delta_n$$

$$P^\delta = P_w \cup \{ x < \delta, x_w \geq \gamma \}$$

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\[ \delta \]

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\[ \delta_0 \quad \delta_1 \quad \delta_2 \quad \ldots \quad \delta \]

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\[ \delta(p) \quad \delta(p_0) \quad \delta(p_2) \]

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(5) Since $P_w \subseteq P_n$ for all $n$:

$$P_w \downarrow x_n = x_n$$

Since $\delta = \sup_n \delta_n$ and $P_w \downarrow \tilde{C}$ is a club in $\tilde{C}$,

so $P_w \downarrow \tilde{\delta} \in \tilde{C}$

(6) By (4) we get

$$P_w \downarrow x_n \in C$$

so $P_w \downarrow x = x_w$

It follows:

(7) $P^\delta \downarrow x \in \tilde{C}$

Now notice (7) says the same thing as (2), or

$$(A_0) \tilde{\delta} = P^\delta(\delta)$$

$\square$
3. Sutchin Tree (2nd) The proof is as follows:

**Conditions**
- $p \triangleleft P$, if $p$ is a countable tree in a set of ordinals below $\omega_1$. Denote the corresponding tree ordering by $\leq_p$. We also require $\omega_1 \leq p$.

**Ordering**
- End-extension: $p \leq q$ if $q$ consists of all levels of $p$ up to $\text{ht}(q)$, and $\leq_q = \leq_p$ restricted to these levels.

**Addition to conditions:**
- $\text{ht}(p)$ is a successor ordinal, and
- For any $q \leq p < \text{ht}(p)$ and $q \in \text{el}(p)$, there exists $t \in \text{cl}(p)$ such that $t \leq_p p'$.

(a) $P$ is $\omega_1$-closed (Essence)
- If $(P_n : n \in \omega)$ is a descending chain:
  - Let $P_\omega = \bigcup P_n$
  - Then obtain $p'$ a lower bound by adding a last level using (2). For any $t \in P_\omega$, add a node $t'$ in the last level.

(b) $\omega_1$ is preserved in the generic extension, i.e., $\omega_1(G) = \omega_1$ whenever $G \in (P,V)$-generic.

(c) For a $(P,V)$-generic $G$ let
  $$T_G = (\omega_1, \bigcup \{ \leq_p | p \in G \})$$
Then $T_G$ is tree on $w_1$. (Exercise.) There is a density argument. Show:

- For each $d \in w_1$

  $$D_d = \{ p \in P \mid d \in \text{ode in } p \}$$

  is dense in $P$.

Next show: $\mathcal{L}_G$ is a tree ordnij (Exercise).

(C) The levels of $T_G$ are countable, as the ordnij of $P$ is by end-extensions.

(d) $ht(T_G) = w_1$. (Exercise) Again a density argument. By induction on $d$, using closure of $P$ for limit $d$, show:

$$E_d = \{ p \in P \mid ht(p) \geq d \}$$

is dense in $P$.

Summary so far:

(a) $T_G$ is an $w_1$-tree in $\mathcal{V}(G)$.

We show: $T_G$ is a Sushin tree in $\mathcal{V}(G)$. 