M281 SPRING 2020 L14

Recall: Our poset $\mathbb{P}$ is as follows:

Conditions are trees of the form $p = (a_p, <_p)$ where $a_p \subseteq \omega_1$ is countable, and such that:

(a) p has a last level, i.e. $ht(p)$ is a successor ordinal. Each node not on the last level is a splitting node.

(b) For every $\alpha < \beta < ht(p)$: If $t \in Lev_\alpha(p)$ then there is some $t' \in Lev_\beta(p)$ s.t. $t <_p t'$.

Ordering $p \leq q \iff p$ end-extends q.

Last time we recorded:

(e) $\mathbb{P}$ is ω_1-closed

If G is $(\mathbb{P}, V)$-generic, we let

$$T_G = (\omega_1, \underbrace{\bigcup \{<_p \mid p \in G\}}_{<_{T_G}})$$

and recorded that in $V[G]$,

(e) T_G is an ω_1-tree

We want to prove:

(f) T_G is a Suslin tree in $V[G]$

For this it suffices to show that every antichain in T_G is countable. Assume $\dot{A}$ is a $\mathbb{P}$-term and $p \in \mathbb{P}$ s.t.

(g) $p \Vdash$ "$\dot{A}$ is a maximal antichain in T_G"

Technical remarks

(i) To make (g) precise, we can use a $\mathbb{P}$-term $\dot{T}$ s.t. $\dot{T}^G = T_G$ for every $(\mathbb{P}, V)$-generic G. Such a term exists by the Maximality principle. Alternatively, we can just plug in the definition of T_G from G; in this case there would be the standard term $\dot{G}$ for the generic in (g).

Starting from (g) we find a condition $p^* \leq p$ s.t.

$$(h) \quad p^* \Vdash \dot{A} \subseteq \check{a}_{p^*}$$

Hence

$$p^* \Vdash \dot{A} \text{ is countable}$$

This can be done for any $q \leq p$, i.e. for any $q \leq p$ we find $q^* \leq q$ s.t. $q^* \Vdash \dot{A}$ is countable. So there are densely many $q^* \leq p$ which force $\dot{A}$ to be countable. It follows that $p \Vdash \dot{A}$ is countable.

<u>To prove (h)</u>: Let $p_0 = p$. We construct a strictly descending chain of conditions $(p_n \mid n \in \omega)$ and an assignment

$$t \longmapsto \zeta_t \quad \text{when } t \in a_n \text{ and } \zeta_t \in a_{n+1}$$

such that

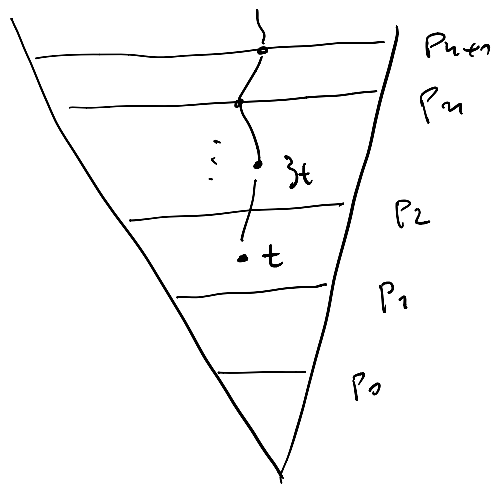
(h1) t, ζ_t are $<_{p_{n+1}}$-comparable

and

(h2) $p_{n+1} \Vdash \check{\zeta}_t \in \dot{A}$

This can be done for each individual $t \in a_n$ using (B) in Forcing Thm for (h2) and extending further the condition if needed to make sure ζ_t is a node in the extended condition. Now we want to have it for all $t \in a_n$. This can be done inductively using the fact that a_n is countable and $\mathbb{P}$ is ω_1-closed.

So we get the following situation:



Let $p_\omega = (a_{p_\omega}, <_{p_\omega})$ where

$$a_{p_\omega} = \bigcup_n a_{p_n}$$

$$<_{p_\omega} = \bigcup_n <_{p_n}$$

<u>Notice</u>: $p_\omega \notin \mathbb{P}$! Because p_ω does not have the last level.

We construct p^* by adding the last level to p_ω in a way that the antichain "denoted" by the term $\dot{A}$ will be sealed off. We do this as follows:

Pick a $t \in a_{p_\omega}$. Now let ζ_t come from (h1) and (h2). So $\zeta_t <_{p_\omega} t$ or $t <_{p_\omega} \zeta_t$. Say the latter applies. By construction we have s_n on the largest level of p_n, where n is smallest s.t. $\zeta_t \in a_{p_n}$, s.t. $\zeta_t < s_n$. Then inductively find s_{k+1} on the largest level of p_{k+1} s.t. $s_k <_{p_{k+1}} s_{k+1}$. This determines a cofinal branch, say b_t through p_ω. If $\zeta_t <_{p_\omega} t$ we proceed analogously. So:

(h3) For each $t \in p_\omega$ we have a cofinal branch b_t through p_ω s.t.

(i) $t \in b_t$

(ii) $\zeta_t \in b_t$

Let p^* be obtained from p_ω by adding one node on top of each b_t. Then (h4) $p^* \in \mathbb{P}$, and $p^* \le p_n$ all n.

By (h2):

$$p^* \Vdash \check{\zeta}_t \in \dot{A}$$

whenever $t \in a_\omega$. Since $\zeta_t \in b_t$ for all $t \in a_\omega$, this tells us:

$$p^* \Vdash \check{b}_t \cap \dot{A} \neq \emptyset$$

Since every node on the last level of p^* is above some b_t, we have:

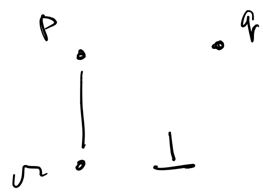
$$p^* \Vdash \text{Every node on the last level of } \check{p}^* \text{ is above some element of } \dot{A}$$

So if H is $(\mathbb{P}, V)$-generic and $p^* \in H$ then every element of the last level of p^* is above some element of $\dot{A}^H$. But the ordering of $\mathbb{P}$ is by end-extension, so the last level of p^* is just $\mathrm{Lev}_{ht(p^*)-1}(T_H)$.
So if $\zeta \in \dot{A}^H$ then $ht_{T_H}(\zeta) < ht(p^*)$ otherwise there is some $\zeta' \in \underbrace{\mathrm{Lev}_{ht(p^*)-1}(T_H)}_{= \text{last level of } p^*}$ s.t. $\zeta' <_{T_H} \zeta$.
But then ζ' was put on top of some b_t so there is $\zeta_t \in b_t$ s.t. $\zeta_t \in \dot{A}^H$ by (h1) and (h2). Hence $\zeta_t <_{T_H} \zeta'$. Contradiction, as $\zeta, \zeta' \in \dot{A}^H$. $\square$ (h)

$\square$ (f)

12.2. Definition A poset $\mathbb{P}$ is <u>separative</u> iff for every $p, q \in \mathbb{P}$ the following holds:

$$p \nleq q \Rightarrow (\exists r \leq p)(r \perp q)$$



The posets $Fn(A, B, \kappa)$ are all separative (easy).

12.3 Definition Let $\mathbb{P}$ be a poset. Define a binary relation $\leq^*$ on $\mathbb{P}$ by

$$p \leq^* q \quad \text{iff} \quad (\forall r \leq p)(r \parallel q)$$

<u>One can show:</u> if $\mathbb{P}$ is separative then $\leq^* = \leq$. Also, $(\mathbb{P}, \leq^*)$ is always separative. Generic extensions by $(\mathbb{P}, \leq^*)$ are the same as with $\mathbb{P}$.

12.4. Definition We say that a poset $\mathbb{P}$ is <u>(κ, λ)-distributive</u> iff generic extensions by $\mathbb{P}$ do not add any functions $f : \gamma \to \lambda$ when $\gamma < \kappa$.
Here we assume κ, λ are cardinals, κ is regular.
$\mathbb{P}$ is <u>(κ, ∞)-distributive</u> iff generic extensions by $\mathbb{P}$ do not add any functions $f : \gamma \to On$ for $\gamma < \kappa$.
We also say "κ-distributive" for (κ, ∞)-distributive.

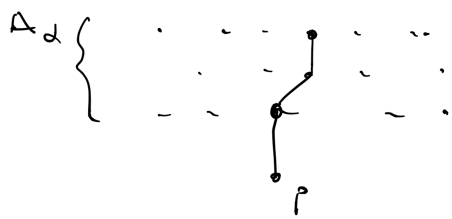
12.5. Theorem (ZFC) Assume κ is regular and $\mathbb{P}$ is a separative poset. TFAE:

(a) $\mathbb{P}$ is κ-distributive

(b) If $(D_\alpha \mid \alpha < \gamma)$ is a sequence of open dense subsets of $\mathbb{P}$ where $\gamma < \kappa$ then $\bigcap_{\alpha<\gamma} D_\alpha$ is a dense subset of $\mathbb{P}$

(c) If $(A_\alpha \mid \alpha < \gamma)$ is a sequence of maximal antichains in $\mathbb{P}$ then there is a condition $p \in \mathbb{P}$ such that

(*) For any $\alpha < \gamma$ there is $q \in A_\alpha$ s.t. $p \leq q$



12.6. Proposition Assume $\mathbb{P}$ is a poset, G is $(\mathbb{P}, V)$-generic and $A \in V$ s.t. $A \subseteq G$. Then there is some $p \in G$ such that

(*) $\quad p \leq^* q$ for all $q \in A$

If $\mathbb{P}$ is separative then we can write "$p \leq q$" in place of "$p \leq^* q$".

Proof Notice:

$$D = \{ p \in \mathbb{P} \mid p \leq^* q \text{ for all } q \in A \text{ or } p \perp q \text{ for some } q \in A \}$$

is dense.

To see this: Pick any $r \in \mathbb{P}$. If $r \leq^* q$ for all $q \in A$ then $r \in D$ and we are done. Otherwise there is some $q \in A$ s.t. $r \not\leq^* q$. By the definition of $\leq^*$ there is some $p \leq r$ s.t. $p \perp q$. Then $p \in D$.

Since D is dense in $\mathbb{P}$: $G \cap D \neq \emptyset$. Let $p \in G \cap D$. Since $A \subseteq G$: $p \parallel q$ for all $q \in A$. So the latter option in the definition of D does not apply, hence $p \leq^* q$ for all $q \in A$. $\square$ P 12.6.

Proof of Thm 12.5

(c) $\Rightarrow$ (b) If $(D_\alpha \mid \alpha < \gamma)$ is a sequence of open dense sets in $\mathbb{P}$, for each $\alpha < \gamma$ pick a maximal anti-chain $A_\alpha \subseteq D_\alpha$. Then if $p \in \mathbb{P}$ satisfies (*) in (c) then $p \in \bigcap_{\alpha<\gamma} D_\alpha$. This shows that $\bigcap_{\alpha<\gamma} D_\alpha \neq \emptyset$. A slight elaboration on this argument actually shows that $\bigcap_{\alpha<\gamma} D_\alpha$ is dense in $\mathbb{P}$ (Exercise.).

(b) $\Rightarrow$ (a). Assume $\dot{f}$ is a $\mathbb{P}$-term and assume

$$\Vdash \dot{f} : \gamma \longrightarrow V$$

(This is not a loss of generality, we may really think that $1_{\mathbb{P}}$ forces this — think about this.)

Now for each $\alpha < \gamma$ let

$$D_\alpha = \{ p \in \mathbb{P} \mid (\exists x \in V) \; p \Vdash \dot{f}(\check{\alpha}) = \check{x} \}$$

Easy to check: D_α is an open dense set in $\mathbb{P}$.

Let
$$D = \bigcap_{\alpha<\gamma} D_\alpha$$
Then for each $p \in D$ and each $\alpha<\gamma$ there is $x_\alpha^p \in V$ s.t.
$$p \Vdash \dot{f}(\check{\alpha}) = \check{x}_\alpha^p$$
So if we define $f^p : \gamma \to V$ by
$$f^p(\alpha) = x_\alpha^p$$
then $p \Vdash \dot{f} = \check{f}^p$

Hence $p \Vdash \dot{f} \in V$. Because this is true for all $p \in D$ and D is dense: $\Vdash \dot{f} \in V$.

$\square$ (b) $\Rightarrow$ (a)