M281 SPRING 2020 L 15

We prove (a) $\Rightarrow$ (c) in Thm 12.5. Recall:

(a) $\mathbb{P}$ is κ-distributive

(c) If $(A_\alpha \mid \alpha < \gamma)$ is a sequence of maximal antichains in $\mathbb{P}$ where $\gamma < \kappa$ then there is a condition $p \in \mathbb{P}$ s.t.

(*) For every $\alpha < \gamma$ there is some $q \in A_\alpha$ such that $p \leq q$.

Notice: because A_α is an antichain, there is exactly one $q \in A_\alpha$ as in (*).

Proof of (a) $\Rightarrow$ (c): Consider a $(\mathbb{P}, V)$-generic G. Then for every $\alpha < \gamma$ the intersection $G \cap A_\alpha$ is a singleton. So in $V[G]$ we have a function $f : \gamma \to \mathbb{P}$ defined by

$f(\alpha)$ = the unique element in the intersection $G \cap A_\alpha$

Since $\mathbb{P}$ is κ-distributive: $f \in V$. (The official definition of κ-distributivity talks about functions $f : \gamma \to On$. But since we work in ZFC, everything applies to functions $f : \gamma \to V$.) So:

$$\operatorname{rng}(f) \subseteq G \quad \text{and} \quad \operatorname{rng}(f) \in V$$

By P 12.6. there is a $p \in G$ s.t. $p \leq q$ for all $q \in \operatorname{rng}(f)$.

□ (a) $\Rightarrow$ (c)

□ Thm 12.5.

12.7. Remark Thm 12.5. has an obvious reformulation for non-separative (or general) posets using $\leq^*$.

12.8. Remark If $\dot{G}$ is the canonical name for the generic filter then TFAE:

(i) $p \leq^* q$

(ii) $p \Vdash \check{q} \in \dot{G}$

(HW problem.)

(4) Shooting a club through a stationary set with initial segments.

Let $S \subseteq \omega_1$ be stationary. Define a poset $\mathbb{P}_S$ as follows.

- Conditions are closed bounded sets $c \subseteq S$. Also let $\alpha_c = \max(c)$ for $c \in \mathbb{P}_S$
- Ordering End-extension.

Let G be $(\mathbb{P}_S, V)$-generic and $C_G = \bigcup G$

(a) C_G is a club in ω_1^V

C_G is closed: Easy

C_G is unbounded in ω_1^V: Because for each $\alpha < \omega_1^V$

$$D_\alpha = \{ c \in \mathbb{P} \mid \alpha_c \geq \alpha \} \quad \text{is dense in } \mathbb{P}$$

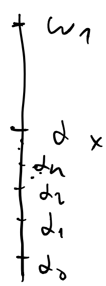
(b) $\mathbb{P}_S$ is ω_1-distributive.

Let $(D_n \mid n \in \omega)$ be a sequence of open dense subsets of $\mathbb{P}$.

Let Θ be large and $X \prec H_\Theta$ s.t.

- $S, \mathbb{P}_S, (D_n \mid n \in \omega) \in X$
- X is countable (Notice $\alpha_X = X \cap \omega_1 \in \omega_1$ for any such X)
- $\alpha_X \in S$ (Possible, as S is stationary see M2PB HW2 Problem 5)

Fix a strictly increasing sequence of ordinals $(\alpha_n \mid n \in \omega)$ such that $\sup_n \alpha_n = \alpha_X$.



Pick some $p \in \mathbb{P} \cap X$. We will find $p^* \leq p$ s.t. $p^* \in \bigcap_{n \in \omega} D_n$.

Now construct a descending chain of conditions.

Since D_0 is dense, there is some $p_0' \leq p$ s.t. $p_0' \in D_0$ and $\alpha_{p_0'} \geq \alpha_0$. Since $p, D_0, \alpha_0, \mathbb{P}_S \in X$ there is such a condition inside X; denote it by p_0. The point of choosing $p_0 \in X$: then $\alpha_{p_0} < \alpha_X$.

Now continue inductively: If we already have $p_n \in X$. Then since D_{n+1} is dense, we have some $p'_{n+1} \leq p_n$ s.t. $p'_{n+1} \in D_{n+1}$. Since $p_n, D_n, \alpha_n, \mathbb{P}_S \in X$ there is such a condition inside X; denote it by p_{n+1}.

So we have constructed a descending chain

$$p_0 \geq p_1 \geq \dots \geq p_n \geq \dots$$

of conditions such that

(i) $p_n \in D_n$ all n

(ii) $\alpha_n \leq lh(p_n) < \alpha_X$

Let

$$p^* = \left(\bigcup_{n\in\omega} p_n\right) \cup \{\alpha_X\}$$

Then $p^* \in \mathbb{P}_S$ because $\alpha_X \in S$.

Since $p^* \leq p_n$ and $p_n \in D_n$ and D_n is open:

$$p^* \in D_n \text{ all } n, \text{ i.e. } p^* \in \bigcap_{n\in\omega} D_n.$$

$\square$ (b)

Notice S may be co-stationary in V, but S is in the club filter from the point of view of $V[G]$, because $C_G \subseteq S$. Equivalently: In $V[G]$: $\omega_1 \setminus S \in NS_{\omega_1}$.

(5) Shooting a club through a stationary set with finite conditions. (Abraham-Shelah).

Let $S \subseteq \omega_1$ be stationary. Define a poset $\mathbb{P}_S$ as follows.

- Conditions: are pairs $p = (I_p, O_p)$ where
  - $I_p \subseteq S$ is finite
  - O_p is a finite set of intervals of the form $(\alpha, \beta]$ (i.e. closed from above!) where $\beta < \omega_1$. Here $\alpha = \beta$ is allowed. We require $I_p \cap (\bigcup O_p) = \emptyset$
- Ordering $p \leq q$ iff $I_p \supseteq I_q$ and $O_p \supseteq O_q$.

Let G be $(\mathbb{P}_S, V)$-generic and
$$C_G = \bigcup \{ I_p \mid p \in G \}$$

(a) $C_G \subseteq S$ and C_G is a club subset of ω_1.

Unboundedness of C_G: Easy density argument

C_G is closed: Point: Let $\xi < \omega_1$. Then
$$D_\xi = \{ p \in \mathbb{P}_S \mid \xi \in I_p \text{ or } \xi \in \bigcup O_p \}$$
is dense in $\mathbb{P}_S$. (Exercise.)

Now assume $\xi \notin C_G$. This means: $\xi \notin I_p$ whenever $p \in G$. This means: if $p \in G \cap D_\xi$ then $\xi \in \bigcup O_p$. So there is an interval $(\alpha, \beta] \in O_p$ s.t. $\xi \in (\alpha, \beta]$. And by the definition of C_G: $(\alpha, \beta] \cap C_G = \emptyset$. Now notice also that
$$(\alpha, \beta] = (\alpha, \beta+1)$$

Conclusion: If $\xi \notin C_G$ then there is an open interval $(\alpha, \beta+1)$ s.t. $\xi \in (\alpha, \beta+1)$ and $(\alpha, \beta+1) \cap C_G = \emptyset$. This proves C_G is closed.

$\square$ C_G closed.

(b) $\omega_1^{V[G]} = \omega_1^V$.

Assume $f \in V[G]$ s.t. $f : \omega \to V$. We show

(1) There is a set $X \in V$, X countable in V such that $\operatorname{rng}(f) \subseteq X$.

Then (1) clearly implies (b).

Fix a $\mathbb{P}_S$-term $\dot{f}$ s.t. $f = \dot{f}^G$. Let $p \in G$ be a condition s.t. $p \Vdash \dot{f} : \check{\omega} \to V$. Let Θ be large and $X \prec H_\Theta$ be s.t.

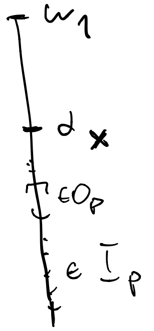
(i) X is countable

(ii) $\alpha_X \in S$

(iii) $\mathbb{P}_S, p, \dot{f} \in X$.

We find a condition $p^* \leq p$ such that for every $n \in \omega$:

$$p^* \Vdash \dot{f}(n) \in X$$



Since $p \in X$: $I_p \subseteq \alpha_X$, $\beta < \alpha_X$ whenever $(\alpha, \beta] \in O_p$. Let

$$p^* = (I_p \cup \{\alpha_X\}, O_p)$$

Then $p^* \in \mathbb{P}_S$ and $p^* \leq p$.

The following holds:

(2) if $q \leq p^*$ then let $q|X = (I_q \cap X, O_q \cap X)$. Then for every $r \in \mathbb{P}_S \cap X$:

$$r \leq q|X \text{ then } r \parallel q$$

<u>Why is this useful:</u> Assume $q \leq p^*$ is s.t.

(3) $H_\theta \models (\exists y)(q \Vdash \dot{f}(\check{n}) = \check{y})$

<u>Notice:</u> $q \leq q|X$. Notice also $q|X \in X$

this is because $I_{q|X} \subseteq X$, the endpoints of intervals in $O_{q|X}$ are in X and $I_{q|X}, O_{q|X}$ are finite.

So we have

(4) $H_\theta \models (\exists r \leq q|X)(\exists y)(r \Vdash \dot{f}(\check{n}) = \check{y})$

Here q from (3) is a witness for the for the $\exists r$ in (4).

Because $X \prec H_\theta$ and $q \restriction X, \dot{f}, n, \mathbb{P}_S \in X$:

There are r, y as in (4) which are inside X.

So we have $r, y \in X$ s.t.

$$H_\theta \models r \Vdash \dot{f}(\check{n}) = \check{y}$$

But by (2): $r \parallel q$

We know $q \Vdash \dot{f}(n) = \check{z}$ for some $\check{z}$

It follows that $q \Vdash \dot{f}(n) = \check{y}$ (i.e. $z = y$).

In particular: $q \Vdash \dot{f}(n) \in \check{X}$.

Now this can be done for densely many $q \leq p^*$, so $p^* \Vdash \dot{f}(n) \in \check{X}$. Also, this can be done for every $n \in \omega$, so $p^* \Vdash \operatorname{rng}(\dot{f}) \subseteq \check{X}$.

This proves (1) modulo (2).