We are discussing the Abraham-Schelbeke poset $P_S$ where $S \subseteq \kappa$ is stationary.

Recall: conditions have the form $p = (I_p, O_p)$ where
- $I_p$ is finite
- $O_p$ is a finite set of intervals of the form $(\alpha, \beta]$ where $\alpha \leq \beta$.
- $I_p \cap (\alpha, \beta] = \emptyset$ whenever $(\alpha, \beta] \in O_p.$

Ordering: $p \leq q$ iff $I_p \subseteq I_q$ and $O_p \subseteq O_q.$

We have the following situation: $X < H$ is countable and such that $2_x \subseteq S$. We have a condition $p \in X$ and we let

\[ p^* = (I_{p^*} \cup \{2_x\}, O_{p^*}) \]

Notice $p^* \in P_S$ as $I_{p^*} \subseteq S$ and $I_{p^*} \cap (\alpha, \beta] = \emptyset$ for all $(\alpha, \beta] \in O_{p^*}$. This is because $O_{p^*} = O_p$ and $2_x \notin X$ whereas $(\alpha, \beta] \subseteq X$ for all $(\alpha, \beta] \in O_p$. We easily have

\[ p^* \leq p. \]

If $p \in X$ then $O_p \subseteq X$, and since $O_p$ is finite: $O_p \subseteq X$ so if $(\alpha, \beta] \in O_p$ then $(\alpha, \beta] \subseteq X$. But $\beta < \omega$, so actually $\beta < 2_x$.

We are proving clause (2):

(2) If $q \in P_S$ is such that $q \leq p^*$, let $q \restriction X = (I_q \cap X, \{(\alpha, \beta] \cap X : \alpha \leq \beta \leq \omega, (\alpha, \beta] \subseteq X, \beta < 2_x\})$ and $q \restriction X$ is called the **reduct or restriction** of $p \to X$. Then for every $r \subseteq P_S \cap X$:

\[ r \leq q \restriction X \implies r \parallel q. \]
So take \( q \leq p \) and look at \( qX \). We show:

\[
(2.1) \quad q' = (I \cap \overline{I}, O \cup Oq) \in P_s
\]

and \( q' \leq n \), \( q' \).

The point is to show that \( q' \in P_s \). That \( q' \leq n \) is obvious.

To see that \( q' \in P_s \) it suffices to show

(i) \( I_\alpha \cap \langle a, \beta \rangle = \emptyset \) whenever \( \langle a, \beta \rangle \in Oq' \)

and

(ii) \( \overline{I}_q \cap \langle a, \beta \rangle = \emptyset \) whenever \( \langle a, \beta \rangle \in O_q \)

Let's do (i) and leave (ii) as an exercise.

Notice \( I \subseteq X \). If \( \langle a, \beta \rangle \in O_q \) then there are two options:

(1) \( \langle a, \beta \rangle \in X \), but then \( \langle a, \beta \rangle \in O_q \) because \( \langle a, \beta \rangle \in O_q \) and \( n \leq qX \), so \( O \supseteq O_q \).

(2) \( \langle a, \beta \rangle \notin X \). Here we use the fact that \( a \in \overline{I}_q \). Thus \( a \in I_q \). Since \( \langle a, \beta \rangle \notin X \), necessary \( \beta \notin X \) but since \( a \in I_q \), necessary \( a \notin \langle a, \beta \rangle \). Hence \( a \leq a \).

It follows that \( \langle a, \beta \rangle \cap X = \emptyset \). Since \( I \subseteq X \), we have \( \langle a, \beta \rangle \cap I = \emptyset \).

\( \Box (2) \)

Remark. The condition \( p^k \) is called a \((P_s, X)\)-generic condition, or \((P_s, X)\)-strongly generic condition.

This is because if \( G \circ (P_s, V) \) generic and \( p \in G \) then

\( G \cap X \circ (P_s | \langle p \cap X \rangle, V) \) - generic where

\( P1 q = \{ p \in P \mid p \leq q \}. \) Any point \( P1 q \in P. \)
12.9. Definition. Let \( x \in \omega \), \( X \) be a set, \( (C, R) \) be a partition of \( X \) (so \( C \cap R = \emptyset \) and \( C \cup R = X \)). The triple \( (\delta, X, (C, R)) \) is called a **game** on \( X \) of length \( \delta \).

The idea behind: We think of two players who cooperate to build a sequence in \( \leq \delta X \). 

- \( C \) stands for "challenger," the player at ordinals in \( C \).
- \( R \) stands for "responder," the player at ordinals in \( R \).

We "play" means that the player picks an element of \( X \).

So a run of the game is of the form

\[
(x_0, x_1, \ldots, x_{\delta}, x_{\delta+1}, \ldots, x_{\delta+\xi})
\]

where \( \delta \leq \xi \), \( x_\gamma \in X \) for all \( \gamma \leq \xi \). Here:

- if \( \gamma \in C \) then \( x_\gamma \) was played by the challenger.
- if \( \gamma \in R \) then \( x_\gamma \) was played by the responder.

A **strategy** for challenger is a function

\[
\sigma : \cup_{\gamma \in C} \delta X \rightarrow X
\]

A **strategy** for responder is a function

\[
\tau : \cup_{\gamma \in R} \delta X \rightarrow X
\]

A payoff set for the game \( (\delta, X, (C, R)) \) is a set \( A \subseteq \leq \delta X \). A game of length \( \delta \) on \( X \) with payoff set \( A \) is generated by

\[
G_{\delta, (C, R)}^X (A)
\]

We say that the responder wins the run \( (x_0, x_1, \ldots, x_{\delta}, x_{\delta+1}, \ldots, x_{\delta+\xi}) \)

if this run is in \( A \).

A strategy \( \tau \) for the responder is **winning** in
\(G^x_{\gamma_1(c,1)}(A)\) iff any run of the game played
by the responder according to \(T\) is in \(A\).

A strategy \(s\) for the challenger is winning on
\(G^x_{\delta_1(c,1)}(A)\) iff any run of the game played by
the challenger according to \(s\) is in \(\subseteq x \times A\).

12.10. Remark (A) In (i) we drop all subscripts
and superscripts if they are clear from the context.

(ii) In practice, we describe games informally, but
with some care they can always be put into the
form as in Def 12.9. In particular, except
the payoff set \(A\) we will also talk about the
rules of the game; these rules can be
however included in the payoff set of \(A\) and
the payoff set accordingly.

12.11. Definition Let \(x \in A^n\) and \(P\) be a poset.
We consider the following game on \(X = P\) of length \(\gamma\).

Player I is the challenger, and plays at odd indices and
Player II is the responder, and plays at all even indices
which are \(> 0\); this includes all limit ordinals.

Rules of the game: if the object played at step \(\gamma\) is
\(\gamma\) is devoted by \(P_\gamma \in P\) then we require
\(P_\gamma \leq P_\gamma\) for all \(\gamma < \gamma\).

In other words, the players alternate to produce a
descending chain in \(P\).

Payoff set: Player II (= responder) wins the run
of the length of the run \(\gamma\) on the
the challenge is the first one to break the rules.

Denote this game by $G(P, \gamma)$.

12.12. Definition Let $\gamma$ be regular and $P$ be a poset.
We say that $P$ is $n$-directed closed off every
directed set $A \subseteq P$ of cardinality $\leq n$ has a lower
bound in $P$, i.e., there is $p \in P$ s.t. $p \leq q$ for all $q \in A$.
Here $A \subseteq P$ is directed off for every $q_1, q_2 \in A$ the $q \in A$ s.t. $q \leq q_1, q_2$. (Easy to see: if $A$ is directed
then every finite subset of $A$ has a lower bound in $A$.)

12.13. Definition Let $P$ be a poset and $\gamma \in \mathbb{N}$. We say
that $P$ is strategically $\gamma$-closed off the responder
(i.e., Player II) has a winning strategy in the game
$G(P, \gamma)$.

12.14. Proposition Let $P$ be a poset and $\gamma$ be regular. Then

$P$ is $n$-directed closed $\Rightarrow$ $P$ is $n$-closed $\Rightarrow$

$\Rightarrow$ $P$ is strategically $n$-closed $\Rightarrow$ $P$ is $n$-distributive.

Proof: The first $\Rightarrow$ is easy, the second two are true because...