M281 SPRING 2020 L16

We are discussing the Abraham–Shelah poset $\mathbb{P}_S$ where $S \subseteq \omega_1$ is stationary.

Recall: conditions have the form $p = (I_p, O_p)$ where

- $I_p \subseteq S$ is finite
- O_p is a finite set of intervals of the form $(\alpha, \beta]$ where $\alpha \leq \beta$.
- $I_p \cap (\alpha, \beta] = \emptyset$ whenever $(\alpha, \beta] \in O_p$

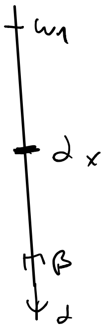
Ordering: $p \leq q$ iff $I_p \supseteq I_q$ and $O_p \supseteq O_q$.

We have the following situation: $X \prec H_\theta$ is countable and such that $\alpha_X \in S$. We have a condition $p \in X$ and we let

$$p^* = (I_p \cup \{\alpha_X\}, O_p)$$

Notice $p^* \in \mathbb{P}_S$ as $I_{p^*} \subseteq S$ and $I_{p^*} \cap (\alpha, \beta] = \emptyset$ for all $(\alpha, \beta] \in O_{p^*}$. This is because $O_{p^*} = O_p$ and $\alpha_X \notin X$ whereas $(\alpha, \beta] \subseteq X$ for all $(\alpha, \beta] \in O_p$. We easily have

$$p^* \leq p.$$



[If $p \in X$ then $O_p \in X$, and since O_p is finite: $O_p \subseteq X$ so if $(\alpha, \beta] \in O_p$ then $(\alpha, \beta] \in X$. But $\beta < \omega_1$ so actually $\beta < \alpha_X$]

We are proving clause (2):

(2) If $q \in \mathbb{P}_S$ is s.t. $q \leq p^*$, let $q \restriction X = (I_p \cap X, O_p \cap X)$
[$q \restriction X$ is called the reduct or restriction of p to X]
Then for every $r \in \mathbb{P}_S \cap X$:
$$r \leq q \restriction X \implies r \parallel q$$

So take $q \le p^*$ and look at $q|X$. We show:

$$(2.1) \qquad q' = (I_r \cup I_q,\ O_r \cup O_q) \in \mathbb{P}_S$$
$$\text{and} \quad q' \le r, q$$

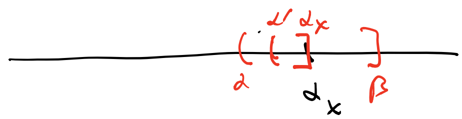
The point is to show that $q' \in \mathbb{P}_S$. That $q' \le r, q$ then, is obvious.

To see that $q' \in \mathbb{P}_S$, it suffices to show

(i) $I_r \cap (\alpha, \beta] = \emptyset$ whenever $(\alpha,\beta] \in O_q$

and (ii) $I_q \cap (\alpha, \beta] = \emptyset$ whenever $(\alpha,\beta] \in O_r$

Let's do (i) and leave (ii) as an exercise.



Intervals $(\alpha,\beta]$ of this kind not allowed. The same for $(\alpha,\alpha_x]$.

Notice $I_r \subseteq X$. If $(\alpha,\beta] \in O_q$ then there are two options:

Either $(\alpha,\beta] \in X$, but then $(\alpha,\beta] \in O_r$ because $(\alpha,\beta] \in O_{q|X}$ and $r \le q|X$, so $O_r \supseteq O_{q|X}$.

Or else $(\alpha,\beta] \notin X$. Here we use the fact that $\alpha_X \in I_{p^*}$. Then $\alpha_X \in I_q$. Since $(\alpha,\beta] \notin X$ necessarily $\beta \notin X$ but since $\alpha_X \in I_q$, necessarily $\alpha_X \notin (\alpha,\beta]$. Hence $\alpha \ge \alpha_X$. It follows that $(\alpha,\beta] \cap X = \emptyset$. Since $I_r \subseteq X$, we have $(\alpha,\beta] \cap I_r = \emptyset$.

$\square$(2)

Remark The condition p^* is called a $(\mathbb{P}_S, X)$-master condition, or a $(\mathbb{P}_S, X)$-strongly generic condition. This is because if G is $(\mathbb{P}_S, V)$-generic and $p^* \in G$ then $G \cap X$ is $(\mathbb{P}_S | p^* \cap X, V)$-generic where

$\mathbb{P}|q = \{p \in \mathbb{P} \mid p \le q\}$ Any poset $\mathbb{P}$, $q \in \mathbb{P}$.

12.9. Definition Let $\gamma \in On$, X be a set (C, R) be a partition of γ (so $C \cap R = \emptyset$ and $C \cup R = \gamma$). The triple $(\gamma, X, (C,R))$ is called a game on X of length γ.

The idea behind: We think of two players who cooperate to build a sequence in ${}^{\leq\gamma}X$.

C stands for "challenger", (s)he plays at ordinals in C

R stands for "responder", (s)he plays at ordinals in R

Here "plays" means that the player picks an element of X.

So a run of the game is of the form

$$(x_\xi \mid \xi < \alpha)$$

where $\alpha \leq \gamma$, $x_\xi \in X$ for all $\xi < \alpha$. Here

- if $\xi \in C$ then x_ξ was played by the challenger
- if $\xi \in R$ then x_ξ was played by the responder.

A strategy for challenger is a function

$$\sigma : \bigcup_{\xi \in C} {}^{\xi}X \longrightarrow X$$

A strategy for responder is a function

$$\tau : \bigcup_{\xi \in R} {}^{\xi}X \longrightarrow X$$

A payoff set for the game $(\gamma, X, (C,R))$ is a set $A \subseteq {}^{\leq\gamma}X$. A game of length γ on X with payoff set A is denoted by

$$(1) \qquad \mathcal{G}^{X}_{\gamma,(C,R)}(A)$$

We say that the responder wins the run $(x_\xi \mid \xi < \alpha)$ iff this run is in A.

A strategy τ for the responder is winning in

$G^X_{\gamma,(C,R)}(A)$ iff any run of the game played by the responder according to τ is in A.

A strategy σ for the challenger is winning in $G^X_{\gamma,(C,R)}(A)$ iff any run of the game played by the challenger according to σ is in ${}^{\leq\omega}X \setminus A$.

12.10. Remark (A) In (1) we drop all subscripts/superscripts if they are clear from the context.

(B) In practice, we describe games informally, but with some work they can always be put into the form as in Def 12.9. In particular, except the payoff set A we will also talk about the rules of the game; these rules can be however included in the payoff set if we amend the payoff set accordingly.

12.11. Definition Let $\gamma \in On$ and $\mathbb{P}$ be a poset. We consider the following game on $X = \mathbb{P}$ of length γ.

Player I is the challenger, and plays at odd ordinals and 0
Player II is the responder, and plays at all even ordinals which are >0; this includes all limit ordinals

Rules of the game: If the object played at step ξ is denoted by $p_\xi \in \mathbb{P}$ then we require
$$p_\xi \leq p_{\bar{\xi}} \quad \text{for all } \bar{\xi} < \xi.$$
In other words, the players alternate to produce a descending chain in $\mathbb{P}$.

Payoff set: Player II (= responder) wins the run if the length of the run is γ or else the challenger is the first one to break the rules.

Denote this game by $G(\mathbb{P}, \gamma)$.

12.12. <u>Definition</u> Let κ be regular and $\mathbb{P}$ be a poset. We say that $\mathbb{P}$ is <u>κ-directed closed</u> iff every directed set $A \subseteq \mathbb{P}$ of cardinality $< \kappa$ has a lower bound in $\mathbb{P}$, i.e. there is $p \in \mathbb{P}$ s.t. $p \leq q$ for all $q \in A$.

Here $A \subseteq \mathbb{P}$ is <u>directed</u> iff for every $q_1, q_2 \in A$ there is $q \in A$ s.t. $q \leq q_1, q_2$. (Easy to see: If A is directed then every finite subset of A has a lower bound in A.)

12.13. <u>Definition</u> Let $\mathbb{P}$ be a poset and $\gamma \in On$. We say that $\mathbb{P}$ is <u>strategically γ-closed</u> iff the responder (i.e. Player II) has a winning strategy in the game $G(\mathbb{P}, \gamma)$.

12.14. <u>Proposition</u> Let $\mathbb{P}$ be a poset and κ be regular. Then

$\mathbb{P}$ is κ-directed closed $\Rightarrow$ $\mathbb{P}$ is κ-closed $\Rightarrow$

$\Rightarrow$ $\mathbb{P}$ is strategically κ-closed $\Rightarrow$ $\mathbb{P}$ is κ-distributive.

<u>Proof</u> The first $\Rightarrow$ is easy, the second two are HW problem