M281 SPRING 2020 L17

12.15. Proposition Assume κ is regular and $\mathbb{P}$ is a strategically κ-closed poset. Then all cardinals and cofinalities $\leq \kappa$ are preserved in all generic extensions by $\mathbb{P}$.
Proof Immediately from P 12.14.

6 Adding a non-reflecting stationary set.

Assume $\kappa > \omega_1$ is regular. We define a poset $\mathbb{P}$ as follows:

- Conditions are pairs $p = (s^p, c^p)$ where
  - $s : \alpha_p + 1 \to \{0,1\}$ where $\alpha_p < \kappa$
  - $c^p = (c^p_\xi \mid \xi \in \lim \cap (\alpha_p + 1))$ where each c^p_ξ is a club in ξ
  - $s(\alpha) = 0$ whenever $\alpha \in c^p_\xi$
- Ordering $p \leq q$ iff $s^p \supseteq s^q$ and $c^p \supseteq c^q$.

(1) $\mathbb{P}$ is strategically κ-closed. So all cardinals and cofinalities $\leq \kappa$ are preserved.

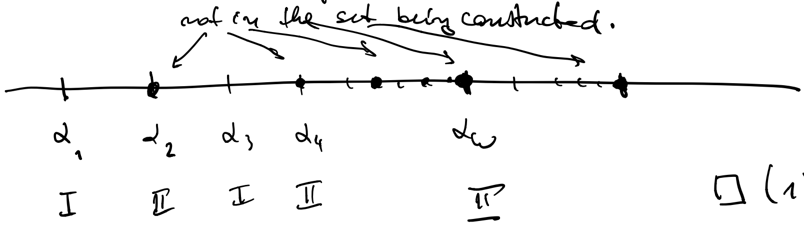
Proof: We construct a winning strategy τ for Player II.

If we already have a run $(p_\beta \mid \beta < \gamma)$ according to τ:

- If γ is a successor ordinal, say $\gamma = \bar{\gamma} + 1$ and it is Player II's turn, Player II just plays any condition $p_\gamma \leq p_{\bar{\gamma}}$ such that $\alpha_\gamma > \alpha_{\bar{\gamma}}$ and $s^{p_\gamma}(\alpha_\gamma) = 0$.
  (To get the control of the endpoints)
- If γ is a limit ordinal: Let
$$\alpha = \sup_{\beta < \gamma} \alpha_\beta$$
then $\alpha > \alpha_\beta$ all $\beta < \gamma$. We let $p_\gamma = (s^{p_\gamma}, c^{p_\gamma})$ where
$$s^{p_\gamma} = \bigcup_{\beta<\gamma} s^{p_\beta} \cup \{\langle \alpha, 0\rangle\}$$
$$c^{p_\gamma} = \bigcup_{\beta<\gamma} c^{p_\beta} \cup \{\langle \alpha, c\rangle\}$$
where $c = \{\alpha_\beta \mid \beta < \gamma \wedge \beta \text{ even}\}$ includes limit ordinals.

So $\alpha_\gamma = \alpha$.

Notice C is a club in α_γ and $s^{p_\gamma}(\alpha)=0$ all $\alpha\in C$, by construction: Player II always plays so that "$s(\alpha)=0$" where α is the last point in the domain.



$\square$ (1)

Let G be $(\mathbb{P},\leq)$-generic,

$$s_G = \bigcup_{p\in G} s^p \qquad c_G = \bigcup_{p\in G} c^p$$

$$S_G = \{\alpha<\kappa \mid s_G(\alpha)=1\}$$

then

(2) $S_G\subseteq\kappa$ and $S_G\cap c^p_\zeta = 0$ for all $\zeta\in\lim\cap\kappa$

This shows: $S_G\cap\zeta$ is non-stationary for all limit $\zeta<\kappa$.

(3) S_G is stationary

Proof Assume $\dot C$ is a $\mathbb{P}$-term and $p\in\mathbb{P}$ s.t.

(3.1) $p\Vdash \dot C$ is a club in $\check\kappa$

We find an ordinal γ and some $p^*\le p$ s.t.

(3.2) $p^*\Vdash \gamma\in\dot C\cap\dot S_G$

This way we get a dense set of $p^*\le p$ for which there exists some γ as in (3.2).

Player I can play $p_1\le p$ and some $\gamma_1>\alpha_p$ s.t.

$$p_1\Vdash\check\gamma_1\in\dot C$$

Then let II play some $p_2\le p_1$ properly extending p_1

II can play according to the winning strategy τ.

Then again let Player I play some $p_3 \leq p_2$ and $\gamma_3 > \alpha_{p_2}$ s.t.

$$p_3 \Vdash \check{\gamma}_3 \in \dot{C}$$

This way construct a descending chain $(p_n \mid n \in \omega)$ and strictly ascending chain $(\gamma_n \mid n \text{ odd})$. Let

$$\gamma = \sup_n \gamma_n$$

Then

$$\gamma = \sup_{n \in \omega} \alpha_{p_n}$$

We construct a condition p^* below all p_n. The

$$(3.3) \quad p^* \Vdash \gamma \in \dot{C}$$

We let

$$p^* = (s^{p^*}, c^{p^*})$$

Guarantees that $\gamma \in S_G$.

where

$$s^{p^*} = \bigcup_{n \in \omega} s^{p_n} \cup \{\langle \gamma, 1 \rangle\}$$

$$c^{p^*} = \bigcup_{n \in \omega} c^{p_n} \cup \{\langle \gamma, c \rangle\}$$

where $c = \{\alpha_{p_n} \mid n \text{ even}\}$ $\square$ (3)

<u>Notice</u>: We can even arrange that γ in (3) has a prescribed cofinality.

(7) <u>Forcing a $\square_\kappa$-sequence</u>

Let $\kappa > \omega$ be a cardinal. The poset $\mathbb{P}$ is defined as follows:

- <u>Conditions</u> $c = (C_\xi \mid \xi < \alpha+1,\ \xi \text{ limit})$ where $\kappa < \alpha < \kappa^+$ such that
  - C_ξ is a club in ξ
  - $C_\xi \cap \bar{\xi} = C_{\bar{\xi}}$ whenever $\bar{\xi} \in \lim(C_\xi)$
  - $\operatorname{otp}(C_\xi) \leq \kappa$
- <u>Ordering</u> $c \leq c'$ iff $c \supseteq c'$

(1) $\mathbb{P}$ is strategically $(\kappa+1)$-closed. So $\mathbb{P}$ is κ^+-distributive, and all cardinals and cofinalities $\leq \kappa^+$ are preserved.

Proof We describe a winning strategy τ for Player II.
Assume we have a run $\langle c^\alpha \mid \alpha < \gamma \rangle$ according to τ and it is Player II's turn.

- If γ is a successor ordinal, say $\gamma = \bar{\gamma}+1$: Player II just plays $c^\gamma \leq c^{\bar{\gamma}}$ which is properly longer.
- If γ is a limit Player II plays

$$c^\gamma = \bigcup_{\bar{\gamma}<\gamma} c^{\bar{\gamma}} \cup \{\langle \alpha, c \rangle\}$$

where $\alpha = \sup_{\bar{\gamma}<\gamma} \alpha_{c_{\bar{\gamma}}}$ and $\alpha_{c_{\bar{\gamma}}} = \text{length}(c_{\bar{\gamma}})$

and

$$c = \{\alpha_{c_{\bar{\gamma}}} \mid \bar{\gamma} < \gamma\}$$

We need to check that c satisfies the conditions for a $\square$-sequence

- c is a club in α by construction (we put all limit points in)
- If $\bar{\alpha}$ is a limit point of c then

$$c \cap \bar{\alpha} = \{\alpha_{c_{\gamma'}} \mid \gamma' < \bar{\gamma}\} \quad \text{for suitable } \bar{\gamma} < \gamma$$

$$(*) \qquad = c_{\bar{\gamma}} \quad \text{Notice this is how we constructed } c_{\bar{\gamma}}$$

& Notice $\gamma' \longmapsto \alpha_{c_{\gamma'}}$ is a normal function

- c^γ is a condition iff $\text{otp}(c) \leq \kappa$. This can be done if $\gamma \leq \kappa$. This is because the function $(*)$ is normal, so $\text{otp}(c) = \gamma$. $\square$

If G is $(\mathbb{P}, V)$-generic then $\bigcup G$ is a $\square_\kappa$-sequence in $V[G]$. This requires to check that length $(\bigcup G) = \kappa^+$, but this is a standard density argument.