M282 SPRING 2020 L18

(8) Forcing a $\square(\kappa)$-sequence; κ regular.

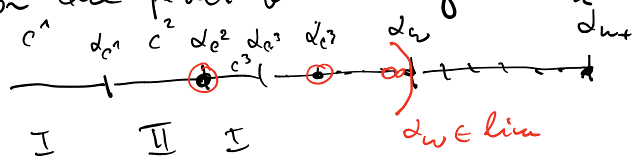
$\mathbb{P}$ is defined as follows:

- Conditions: sequences $c = (c_\xi \mid \xi \leq \alpha_c, \xi \text{ limit})$ s.t.
  - c_ξ is a club subset of ξ
  - if $\bar\xi \in \lim(c_\xi)$ then $c_{\bar\xi} = c_\xi \cap \bar\xi$
- Ordering end-extension: $c \leq c'$ iff $c \supseteq c'$.

(1) $\mathbb{P}$ is strategically κ-closed.

Proof Like the proof of strategic $(\kappa+1)$-closure for the poset for adding a $\square_\kappa$-sequence.

$c^1 \geq c^2 \geq c^3 \geq \cdots \geq c^\gamma \cdots$



$c^{\omega+\omega} \cap \alpha_\omega = \{\alpha_\xi \mid \xi < \omega, \xi \text{ even}\} = c^{\omega+\omega}_\omega$.

- At successor steps II responds with a proper extension
- At limit steps γ let $\alpha_\gamma = \sup_{\xi<\gamma} \alpha_\xi$

  II plays c when

$$c^\gamma = \bigcup_{\bar\gamma<\gamma} c^{\bar\gamma} \cup \{\langle \alpha_\gamma, c\rangle\}$$

where $c = \{\alpha_\xi \mid \xi < \gamma \wedge \xi \text{ even}\}$

To see that $c^\gamma \in \mathbb{P}$: The main part is to check the coherence. (The sets on the c^γ are clubs by construction, routine.). But if $\bar\xi \in \lim(c)$ then $\bar\xi = \alpha_{\bar\gamma}$ for some limit $\bar\gamma < \gamma$. Then

$$c \cap \bar\xi = c \cap \alpha_{\bar\gamma} = \{\alpha_\eta \mid \eta < \bar\gamma \wedge \eta \text{ even}\} = c^\gamma_{\alpha_{\bar\gamma}}$$

$\square$

If G is $(\mathbb{P}, V)$-generic then
$(c^G_\xi \mid \xi \in \lim \cap \kappa^+) = \bigcup G$

is a coherent sequence, i.e.

- C^G_ξ is a club in ξ
- if $\bar\xi \in \lim(C^G_\xi)$ then $C_{\bar\xi} = C_\xi \cap \bar\xi$.

(2) The sequence $(C^G_\xi \mid \xi \in \lim \cap \kappa^+)$ does not have a thread in $V[G]$.

Proof: Assume $\dot C$ is a $\mathbb{P}$-term and $p \in \mathbb{P}$ s.t.

$$p \Vdash \dot C \text{ is a club in } \check\kappa$$

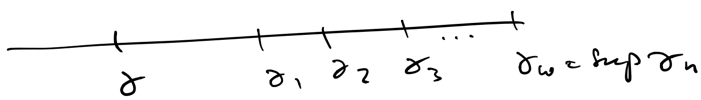
We can find $\gamma < \kappa$ and $p_0 \le p$ s.t.

$$p_0 \Vdash \check\gamma \in \dot C$$

Now inductively construct a descending sequence of conditions $(p_n \mid n \in \omega)$ and a strictly ascending sequence of ordinals $(\gamma_n \mid n \in \omega)$ s.t.

(i) $\gamma_n, \alpha_{p_n} < \gamma_{n+1}, p_{n+1}$ and $\gamma_0 = \gamma$

(ii) $p_n \Vdash \gamma_n \in \dot C$



Now construct $p^* \le p_n$ by letting

$$p^* = \bigcup_{n \in \omega} p_n \cup \{\langle \gamma_\omega, \{\gamma_n \mid n \in \omega \setminus \{0\}\} \rangle\}$$

Then $p^* \Vdash \gamma_n \in \dot C$ all $n \in \omega$, so since

$$p^* \Vdash \dot C \text{ is a club in } \check\kappa$$

we also have

$$p^* \Vdash \gamma_\omega \in \lim(\dot C)$$

Notice also that

$$p^* \Vdash \dot C^G_{\check\gamma_\omega} = \{\gamma_n \mid n \in \omega \setminus \{0\}\}\check{}$$

So since $p^* \Vdash \check{\gamma} \in \dot{C}$, we have

$$p^* \Vdash \dot{C} \cap \check{\gamma}_\omega \neq \check{C}^{G}_{\gamma_\omega}$$

It follows:

$$p^* \Vdash \dot{C} \text{ is not a thread through } (\dot{C}_\zeta \mid \zeta \in \lim \cap \kappa^+).$$

Now we can get this way a dense set of such p^*. $\square$

(9) Assume $(a_\alpha \mid \alpha < \kappa)$ is a $\diamondsuit_\kappa$-sequence where κ is regular and $\mathbb{P}$ is a κ-closed poset. Then $(a_\alpha \mid \alpha < \kappa)$ is a $\diamondsuit_\kappa$-sequence in any generic extension via $\mathbb{P}$.

Proof. Suppose not. Then we can find $\mathbb{P}$-terms $\dot{A}, \dot{C}$ and some $p \in \mathbb{P}$ s.t.

$$p \Vdash \dot{A} \subseteq \check{\kappa} \wedge \dot{C} \text{ is a club in } \check{\kappa} \wedge \dot{A} \cap \alpha \neq \check{a}_\alpha \text{ all } \alpha \in \dot{C}$$

Using the κ-closure of $\mathbb{P}$, inductively construct a descending chain of conditions $(p_\beta \mid \beta < \kappa)$ and a normal sequence of ordinals $(\gamma_\beta \mid \beta < \kappa)$ s.t.

- $p_\beta \Vdash \check{\gamma}_\beta \in \dot{C}$
- For each $\zeta < \gamma_\beta$: $p_\beta \Vdash \check{\zeta} \in \dot{A}$ or $p_\beta \Vdash \zeta \notin \dot{A}$

Then if we let

$$A = \{\zeta < \kappa \mid p_\beta \Vdash \check{\zeta} \in \dot{A} \text{ some } \beta\}$$

$$C = \{\gamma_\beta \mid \beta < \kappa\}$$

Then $A, C \in V$, $A \subseteq \kappa$, C is a club in κ and

$$A \cap \gamma_\beta \neq a_{\gamma_\beta} \text{ all } \beta < \kappa$$

which witnesses that $(a_\alpha \mid \alpha < \kappa)$ is not a $\diamondsuit_\kappa$-sequence in V. Contradiction. $\square$

(10) Vopěnka Algebra

Theorem (ZF) (Vopěnka late 60's) Every set of ordinals is set-generic over HOD. In fact (Woodin) if x is a set of ordinals then

$$HOD_{\{x\}} = HOD[G_x]$$

where G_x is a filter generic for Vopěnka algebra over HOD.

Hence, if $V \models ZFC$ then every $x \in V$ is set-generic over HOD.

Recall if $\mathbb{B}$ is a Boolean algebra we view $\mathbb{B}^+ = \mathbb{B} \setminus \{0_{\mathbb{B}}\}$ as a forcing poset. But we often abuse the notation and write $\mathbb{B}$ in place of $\mathbb{B}^+$.

Assume $\alpha \in On$ and $x \in \mathcal{P}(\alpha)$. We define a Boolean algebra $\mathbb{B}^*$ as follows:

- $A \in \mathbb{B}^* \iff A \in \mathcal{P}(\mathcal{P}(\alpha)) \cap OD$
- Ordering: $A \leq B$ iff $A \subseteq B$

Notice: $\mathbb{B}^*$ is OD-complete, i.e. if $(A_\xi \mid \xi < \beta)$ is a sequence of elements of $\mathbb{B}^*$ and this sequence is OD then $\bigcup_{\xi<\beta} A_\xi, \bigcap_{\xi<\beta} A_\xi \in \mathbb{B}^*$.

Fix a definable

$$\pi : OD \longrightarrow {}^{<\omega}On$$

which to each $z \in OD$ assigns its code. Recall $(\alpha, \varphi, \gamma_1 \dots \gamma_k)$ is a code for z iff z is the unique object u s.t. $V_\alpha \models \varphi(u, \gamma_1 \dots \gamma_k)$. Here we code φ as a member of ω.

So for instance the map π defined by

$$\pi(z) = \text{the } <^* \text{ least code of } z$$

is an example of such a definable bijection.

We let

$$\mathbb{B} = \pi[\mathbb{B}^*]$$

and
$$\leq_{\mathbb{B}} = \pi[\subseteq \cap (\mathbb{B}^* \times \mathbb{B}^*)]$$

then $(\mathbb{B}, \leq_{\mathbb{B}}) \in HOD$ and

$$\pi \restriction \mathbb{B}^* : (\mathbb{B}^*, \subseteq) \to (\mathbb{B}, \leq_{\mathbb{B}})$$

is an OD isomorphism between $(\mathbb{B}^*, \subseteq)$ and $(\mathbb{B}, \leq_{\mathbb{B}})$.

In what follows we work either with $\mathbb{B}$ or $\mathbb{B}^*$ (but mainly $\mathbb{B}^*$) and understand that everything can be transferred to the "other side" via π.

$\mathbb{B}$ or $\mathbb{B}^*$ is called the Vopěnka algebra.

(1) Assume G^* is OD-generic for $\mathbb{B}^*$, i.e. $G^* \cap D \neq \emptyset$ whenever $D \in OD$ is dense in $\mathbb{B}^*$. Then $\bigcap G^*$ is a singleton.

<u>Proof</u> For each $\xi < \alpha$ let

$$A^0_\xi = \{a \in \mathcal{P}(\alpha) \mid \xi \notin a\}$$

$$A^1_\xi = \{a \in \mathcal{P}(\alpha) \mid \xi \in a\}$$

then $A^0_\xi \cup A^1_\xi = \mathcal{P}(\alpha) \in G^*$ and $A^0_\xi \cap A^1_\xi = \emptyset$.
So for each $\xi < \alpha$ exactly one of A^0_ξ, A^1_ξ is in G^*.

Letting

$$y = \{\xi < \alpha \mid A^1_\xi \in G^*\}$$

we get

$$\bigcap G^* = \{y\} \qquad \Box$$

(2) Recall we fixed $x \in \mathcal{P}(\alpha)$. Define

$$G^*_x = \{A \in \mathbb{B}^* \mid x \in A\}$$

Then G^*_x is an OD-generic filter for $\mathbb{B}^*$.

<u>Proof</u> Obviously G^*_x is a filter.

OD-genericity: Assume $D \in OD$ is dense in $\mathbb{B}^*$. Then $\bigcup D = \mathcal{P}(\alpha)$; otherwise $\mathcal{P}(\alpha) \setminus \bigcup D \neq \emptyset$ but $\mathcal{P}(\alpha) \setminus \bigcup D \in \mathbb{B}^*$, so there would be some $A \in D$ (by density) s.t. $A \subseteq \mathcal{P}(\alpha) \setminus \bigcup D$. Contradiction. Since $\bigcup D = \mathcal{P}(\alpha)$ there must be some $A \in D$ s.t. $x \in A$. This means: $D \cap G^*_x \ni A$. $\Box$

Now by (1): $\{x\} = \bigcap G^*_x$. Now everything can be transferred to HOD via π. We let

$$G_x = \pi[G^*_x]$$

and show $x \in HOD[G_x]$. We actually now show

(3) $$HOD_{\{x\}} = HOD[G_x].$$

The inclusion $\supseteq$ is easy, as $HOD \subseteq HOD_{\{x\}}$ and $G_x \in HOD_{\{x\}}$.

To see $HOD_{\{x\}} \subseteq HOD[G_x]$:

Since $HOD_{\{x\}} \models ZFC$ it suffices to show that if $y \subseteq \beta$ for some $\beta \in On$ then $y \in HOD_{\{x\}} \Rightarrow y \in HOD[G_x]$.
So consider such a $y \in HOD_{\{x\}}$. This means we have a formula $\varphi(u, v, w_1 \dots w_k)$ and ordinals $\gamma_1 \dots \gamma_k$ s.t.

For every $\zeta < \beta$:

$$(3.1) \quad \zeta \in y \iff V \models \varphi(\zeta, x, \gamma_1 \dots \gamma_k)$$

Let

$$A^*_\zeta = \{z \in \mathcal{P}(\alpha) \mid V \models \varphi(\zeta, z, \gamma_1 \dots \gamma_k)\}$$

and notice that $A^*_\zeta \in \mathbb{B}^*$, and by (3.1)

$$\zeta \in y \iff x \in A^*_\zeta \iff A_\zeta \in G^*_x$$

So we can write a $\mathbb{B}^*$-term $\dot{y}^*$ for y:

$$\dot{y}^* = \{\langle A^*_\zeta, \check{\zeta} \rangle \mid \zeta < \beta\}$$

Then $\dot{y}^{*G^*_x} = y$. Now using the isomorphism π we can transfer everything to HOD and get a $\mathbb{B}$-term $\dot{y} \in HOD$ s.t.

$$\dot{y}^{G_x} = \dot{y}^{*G^*_x} = y. \qquad \square \, (3)$$

The above is just a basic form of Vopěnka algebra and has many variations. E.g. one can add $\mathbb{R}^n$ to HOD.