MATH 281 SPRING 2020 L19

13. EMBEDDINGS, PROJECTIONS AND AUTOMORPHISMS

13.1. Def Let $\mathbb{P}, \mathbb{Q}$ be posets. A map $e: \mathbb{P} \to \mathbb{Q}$ is called an *embedding* iff for all $p, q \in \mathbb{P}$:

(a) $p \leq q \implies e(p) \leq e(q)$

(b) $p \perp q \implies e(p) \perp e(q)$

13.2. Def An embedding $e: \mathbb{P} \to \mathbb{Q}$ is *regular* iff for every maximal antichain A in $\mathbb{P}$, the pointwise image $e[A]$ is a maximal antichain in $\mathbb{Q}$.

13.3. Remark Some authors use the term "complete" embedding instead (Kunen). The motivation comes from complete Boolean algebras: If $\mathbb{B}, \mathbb{C}$ are complete Boolean algebras then a homomorphism $h: \mathbb{B} \to \mathbb{C}$ is a complete homomorphism (i.e. $h(\bigvee X) = \bigvee h[X]$ all $X \subseteq \mathbb{B}$) iff $h \restriction \mathbb{B}^+ : \mathbb{B}^+ \to \mathbb{C}^+$ is a regular embedding.
Note: An antichain $A \subseteq \mathbb{B}$ is maximal iff $\bigvee A = 1_{\mathbb{B}}$.

13.4. Def Let $e: \mathbb{P} \to \mathbb{Q}$ be an embedding of two posets and $q \in \mathbb{Q}$. An element $p \in \mathbb{P}$ is called a *reduct* of q w.r.t. e iff for every $p' \leq p$: $e(p') \parallel q$.

13.5. Proposition Let $e: \mathbb{P} \to \mathbb{Q}$ be an embedding of posets. TFAE

(a) e is regular

(b) For every predense $B \subseteq \mathbb{P}$: $e[B]$ is predense in $\mathbb{Q}$

(c) Every element $q \in \mathbb{Q}$ has a reduct in $\mathbb{P}$ w.r.t. e.

Proof (a) $\Rightarrow$ (b) Fix a predense $B \subseteq \mathbb{P}$. Then

$$D = B\downarrow = \{ p \in \mathbb{P} \mid p \leq p' \text{ for some } p' \in B \}$$

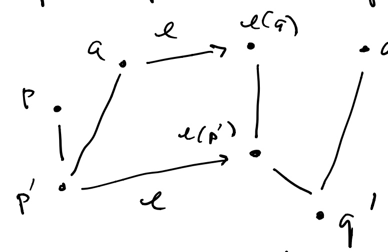
Pick some antichain $A \subseteq D$ which is maximal with respect to $\subseteq$. Because D is dense in $\mathbb{P}$, A is a maximal antichain in $\mathbb{P}$. Since e is regular, $e[A]$ is a maximal antichain in $\mathbb{Q}$. Now if $q \in \mathbb{Q}$ then $q \parallel e(a)$ for some $a \in A$. But $a \leq p$ for some $p \in B$ by definition. So $e(a) \leq e(p)$. Because $q \parallel e(a)$ also $q \parallel e(p)$. So every $q \in \mathbb{Q}$ is compatible with some element of $e[B]$, i.e. $e[B]$ is predense in $\mathbb{Q}$.

(b) $\Rightarrow$ (c). By contradiction. Assume there is $q \in \mathbb{Q}$ which does not have a reduct in $\mathbb{P}$ w.r.t. e. This means that below every $p \in \mathbb{P}$ there is some $p' \leq p$ s.t. $e(p') \perp q$. In other words: the set

$$D = \{p' \in \mathbb{P} \mid e(p') \perp q\}$$

is dense in $\mathbb{P}$. By (b): $e[D]$ is predense in $\mathbb{Q}$, so there must be some $p' \in D$ s.t. $e(p') \parallel q$. Contradiction.

(c) $\Rightarrow$ (a) Let A be a maximal antichain in $\mathbb{P}$. Since e is an embedding, $e[A]$ is an antichain in $\mathbb{Q}$. We need to see: $e[A]$ is maximal. Pick $q \in \mathbb{Q}$. Let $p \in \mathbb{P}$ be a reduct of q w.r.t. e. Because A is maximal, there is some $a \in A$ s.t. $a \parallel p$. Let $p' \leq a, p$. Since p is a reduct and $p' \leq p$: $e(p') \parallel q$. Since $p' \leq a$: $e(p') \leq e(a)$. Hence $e(a) \parallel q$.



This shows q is compatible with some element of $e[A]$, hence that $e[A]$ is maximal. $\square$ (P 13.5)

<u>13.6. Remark</u> Clause (c) in P 13.5. is important because it gives a <u>first-order</u> characterization of regularity.

<u>13.7. Proposition</u> Let $e : \mathbb{P} \to \mathbb{Q}$ be a regular embedding. If H is a $(\mathbb{Q}, V)$-generic filter then $G = e^{-1}[H]$ is a $(\mathbb{P}, V)$-generic filter.

<u>Proof</u> Let $G = e^{-1}[H]$. Then G is upward closed and any two elements of G are compatible. Both

follow from the fact that H is a filter: the former follows because e is order-preserving, the latter follows because e is incompatibility preserving.

Next: $G \cap A \neq \emptyset$ for every maximal antichain in $\mathbb{P}$. This is because $e[A]$ is a maximal antichain in $\mathbb{Q}$, so $H \cap e[A] \neq \emptyset$. So if $p \in A$ is s.t. $e(p) \in H$ then $p \in A \cap G$.

From this we get that G is a filter, i.e. any two elements $p_1, p_2 \in G$ have a lower bound in G. This follows by a simple density argument (Exercise.) $\square$ 13.7.

13.8. Def An embedding $e: \mathbb{P} \to \mathbb{Q}$ is dense iff $\mathrm{rng}(e)$ is a dense subset of $\mathbb{Q}$.

13.9. Proposition Any dense embedding is a regular embedding. (Exercise)

13.10. Proposition If $e: \mathbb{P} \to \mathbb{Q}$ is a dense embedding and G is $(\mathbb{P}, V)$-generic then $H = e[G]\uparrow$ is $(\mathbb{Q}, V)$-generic.

Proof For any filter $G' \subseteq \mathbb{P}$: $e[G']$ is a filter basis, so $H = e[G]\uparrow$ is a filter on $\mathbb{Q}$. To see that H is $(\mathbb{Q}, V)$-generic, it suffices to see that $H \cap D \neq \emptyset$ for every open dense $D \subseteq \mathbb{Q}$. For this it suffices to show that $e^{-1}[D]$ is dense whenever $D \subseteq \mathbb{Q}$ is open dense. (Exercise.) $\square$ 13.10.

13.11. Corollary If $e: \mathbb{P} \to \mathbb{Q}$ is a dense embedding then for every G which is $(\mathbb{P}, V)$-generic there is a $(\mathbb{Q}, V)$-generic H s.t. $V[G] = V[H]$ and for every $(\mathbb{Q}, V)$-generic H there is $(\mathbb{P}, V)$-generic G s.t. $V[H] = V[G]$.

Proof $G = e^{-1}(H)$ and $H = e[G]\uparrow$. $\square$

13.12. Def A map $e: \mathbb{P} \to \mathbb{Q}$ is an isomorphism iff
(i) e is a bijection
(ii) for every $p_1, p_2 \in \mathbb{P}$:
$$p_1 \leq p_2 \iff e(p_1) \leq e(p_2)$$

An isomorphism $e: \mathbb{P} \to \mathbb{P}$ is called an automorphism of $\mathbb{P}$.

13.13 Prop A surjective embedding is a dense embedding. Any isomorphism is a dense embedding. □

13.14. Def Let $e: \mathbb{P} \to \mathbb{Q}$ be a map. We define a map

$$e^*: V^{\mathbb{P}} \to V^{\mathbb{Q}}$$

as follows:

$$e^*(\dot{x}) = \{(e(p), e^*(\dot{z})) \mid (p, \dot{z}) \in \dot{x}\}$$

Notice that this is a definition by $\in$-recursion.

13.15. Proposition Let $e: \mathbb{P} \to \mathbb{Q}$ be a regular embedding, H be $(\mathbb{Q}, V)$-generic and $G = e^{-1}[H]$. Then

$$\dot{x}^G = e^*(\dot{x})^H$$

Proof
$$e^*(\dot{x})^H = \{(e(p), e^*(\dot{z})) \mid (p, \dot{z}) \in \dot{x}\}^H$$
$$= \{e^*(\dot{z})^H \mid (p, \dot{z}) \in \dot{x} \wedge e(p) \in H\}$$
$$= \{\dot{z}^G \mid (p, \dot{z}) \in \dot{x} \wedge p \in G\} = \dot{x}^G \quad \square$$

13.16. Proposition $e^*(\check{a}) = \check{a}$ (Exercise)

13.17. Proposition Let $e: \mathbb{P} \to \mathbb{Q}$ be an embedding, $\varphi(v_1 \ldots v_k)$ be a formula and $\dot{x}_1 \ldots \dot{x}_k$ be $\mathbb{P}$-terms. Let $p \in \mathbb{P}$.

(a) If e is regular and φ is a Δ_0 Σ_1 – formula then

$$p \Vdash_{\mathbb{P}} \varphi(\dot{x}_1 \ldots \dot{x}_k) \implies e(p) \Vdash_{\mathbb{Q}} \varphi(e^*(\dot{x}_1), \ldots, e^*(\dot{x}_k))$$

(b) If e is a dense embedding then the above implication is true for any formula φ.

Proof Let $H \ni e(p)$ be $(\mathbb{Q}, V)$-generic. Then $G = e^{-1}[H]$ is $(\mathbb{P}, V)$-generic and $p \in G$. Since $p \Vdash_{\mathbb{P}} \varphi(\dot{x}_1 \ldots \dot{x}_k)$, we have $V[G] \models \varphi(\dot{x}_1^G, \ldots, \dot{x}_k^G)$. But $G \in V[H]$, so $V[G] \subseteq V[H]$

Since φ is $\overset{\Sigma_1}{\Delta_0}$, by absoluteness we have $V[H] \models \varphi(\dot{x}_1^G \dots \dot{x}_k^G)$.
But now $\dot{x}_i^G = e^*(\dot{x}_i)^H$ by P 13.15, so $V[H] \models \varphi(e^*(\dot{x}_1)^H \dots e^*(\dot{x}_k)^H)$.
We conclude $e(p) \Vdash_{\mathbb{Q}} \varphi(e^*(\dot{x}_1), \dots e^*(\dot{x}_k))$. $\square$ (a)

Regarding (b): the same works, but now we have $V[G] = V[H]$ so the step from $V[G]$ to $V[H]$ is trivial.

$\square$(P13.17)

13.18. Def A poset $\mathbb{P}$ is weakly homogeneous iff for any $p, q \in \mathbb{P}$ there is an automorphism π of $\mathbb{P}$ s.t.

$$\pi(p) \parallel q$$

13.18. Prop $\mathbb{P} = \mathrm{Fn}(A, B, \kappa)$ is weakly homogeneous via an automorphism π s.t. $\pi^2 = id$.

Proof Fix $p, q \in \mathbb{P}$. Let

$$d = \{ a \in A \mid p(a) \neq q(a) \}$$

For $r \in \mathbb{P}$ then let $\pi(r) \in \mathbb{P}$ be s.t.

$$\pi(r)(a) = \begin{cases} q(a) & \text{if } r(a) = p(a) \text{ and } a \in d \\ p(a) & \text{if } r(a) = q(a) \text{ and } a \in d \\ r(a) & \text{if otherwise} \end{cases}$$

$\square$ (P13.18)

13.19. Proposition Assume $\mathbb{P}$ is weakly homogeneous, $A \in V$, G is $(\mathbb{P}, V)$-generic and $X \subseteq A$ is s.t.

(a) $X \in V[G]$

(b) X is definable in $V[G]$ from parameters in V

Then $X \in V$.

Proof Assume $\varphi(u, v_1 \dots v_k)$ is a formula and $b_1 \dots b_k \in V$ such that

$$V[G] \models (\forall a \in A)(a \in X \leftrightarrow \varphi(a, b_1 \dots b_k))$$

For $a \in X$:

$$V[G] \models \varphi(a, b_1 \dots b_k) \quad \text{, i.e.}$$
$$V[G] \models \varphi(\check{a}^G, \check{b}_1^G, \dots \check{b}_k^G)$$

By Forcing Thm (B) there is some $p \in G$ s.t.

(1) $\quad p \Vdash \varphi(\check{a}, \check{b}_1 \dots \check{b}_k)$

Now consider any $q \in \mathbb{P}$. Since $\mathbb{P}$ is weakly homogeneous, we have an automorphism $\pi : \mathbb{P} \to \mathbb{P}$ s.t. $\pi(p) \parallel q$. By applying π to (1):

$$\pi(p) \Vdash \varphi(\pi(\check{a}), \pi(\check{b}_1) \dots \pi(\check{b}_k))$$

Hence
$$\pi(p) \Vdash \varphi(\check{a}, \check{b}_1, \dots \check{b}_k)$$

by p 13.16. So if we take $q' \le \pi(p), q$ then
$$q' \Vdash \varphi(\check{a}, \check{b}_1, \dots, \check{b}_k).$$

Since q was taken arbitrary, we have proved:
$$D = \{ q' \in \mathbb{P} \mid q' \Vdash \varphi(\check{a}, \check{b}_1, \dots \check{b}_k) \}$$
is dense in $\mathbb{P}$. That is,
$$\Vdash \varphi(\check{a}, \check{b}_1, \dots \check{b}_k)$$

So we have:
$$a \in X \implies \Vdash_{\mathbb{P}} \varphi(\check{a}, \check{b}_1, \dots \check{b}_k)$$

Similarly we get
$$a \notin X \implies \Vdash_{\mathbb{P}} \neg\varphi(\check{a}, \check{b}_1, \dots \check{b}_k)$$

So $X = \{ a \in A \mid \Vdash_{\mathbb{P}} \varphi(\check{a}, \check{b}_1 \dots \check{b}_k) \} \in V$ $\square$