MATH 281 SPRING 2020 L 20

13.20. Remark Regarding strong master conditions from the discussion of Abraham – Shelah forcing: Recall: if $\mathbb{P}$ is a poset and $p \in \mathbb{P}$ then

$$\mathbb{P}/p = \{ p' \in \mathbb{P} \mid p' \leq p \} \quad \text{with the same ordering}$$

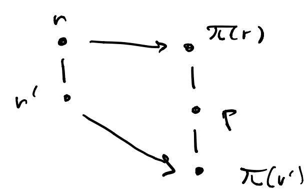
$$(\text{or} \quad \{ p' \in \mathbb{P} \mid p' \parallel p \})$$

One can check: p is a strong master condition for $\mathbb{P}$ w.r.t X iff $id \restriction (\mathbb{P}/p \cap X) \longrightarrow \mathbb{P}/p$ is a regular embedding.

13.21. Definition Assume $\mathbb{P}, \mathbb{R}$ are posets. A map $\pi : \mathbb{R} \to \mathbb{P}$ is a *projection* iff the following hold:

(a) $r \leq r' \Rightarrow \pi(r) \leq \pi(r')$

(b) For every $r \in \mathbb{R}$ and every $p \leq \pi(r)$ there is an $r' \leq r$ s.t. $\pi(r') \leq p$



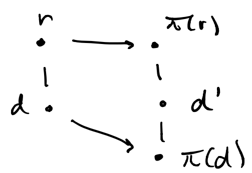
In other words: $\pi[r\downarrow]$ is dense below $\pi(r)$.

13.22. Prop Assume $\pi : \mathbb{R} \longrightarrow \mathbb{P}$ is a projection.

(a) If $D \subseteq \mathbb{P}$ is open dense then $\pi^{-1}[D]$ is dense in $\mathbb{R}$

(b) If H is $(\mathbb{R}, V)$-generic then $G = \pi[H]\uparrow$ is $(\mathbb{P}, V)$-generic

Proof (b) is a direct consequence of (a). We prove (a). Pick $r \in \mathbb{R}$



Since D is dense in $\mathbb{P}$ we have some $d' \in D$ s.t. $d' \leq \pi(r)$. Since π is a projection there is some $d \leq r$ s.t. $\pi(d) \leq d'$

Because D is open: $\pi(d) \in D$. So $d \leq r$ and $d \in \pi^{-1}[D]$. $\square$

13.23. Definition Let $\pi : \mathbb{R} \longrightarrow \mathbb{P}$ be a projection and G be $(\mathbb{P}, V)$-generic. We define the quotient of $\mathbb{R}$ by G with respect to π, denoted by $\mathbb{R}/(G,\pi)$ as follows

- Conditions: elements of $\pi^{-1}[G]$
- Ordering: the restriction of $\leq_{\mathbb{R}}$ to $\pi^{-1}[G]$.

Similarly we can define quotient in terms of regular embeddings:

13.24. Definition If $e : \mathbb{P} \longrightarrow \mathbb{R}$ is a regular embedding and G is $(\mathbb{P}, V)$-generic then we define the quotient of $\mathbb{R}$ by G with respect to e, denoted by $\mathbb{R}/(G,e)$ as follows:

- Conditions are all conditions $r \in \mathbb{R}$ s.t.
  $$r \parallel \pi(p) \quad \text{for all } p \in G$$
- Ordering is the restriction of $\leq_{\mathbb{R}}$ to the set of conditions in the quotient.

13.25 Theorem Assume $\pi : \mathbb{R} \longrightarrow \mathbb{P}$ is a projection.

(a) If G is $(\mathbb{P}, V)$-generic and H is $(\mathbb{Q}, V[G])$-generic where $\mathbb{Q} = \mathbb{R}/(G,\pi)$ then H is $(\mathbb{R}, V)$-generic and $G = \pi[H]\uparrow$.

(b) If H is $(\mathbb{R}, V)$-generic and $G = \pi[H]\uparrow$ then H is $(\mathbb{R}/(G,\pi), V[G])$-generic.

Proof (a) Obvious: $H \subseteq \mathbb{R}$ is a filter. To see genericity:
Let $D \in V$ be (open) dense in $\mathbb{R}$. We show:

(1) $D \cap \mathbb{Q}$ is a dense subset of $\mathbb{Q}$.

Given (1): Since H is $(\mathbb{Q}, V[G])$-generic and $D \cap \mathbb{Q} \in V[G]$ we have $\emptyset \neq H \cap (D \cap \mathbb{Q}) \subseteq H \cap D$.

$q \longrightarrow \pi(q)$

To see (1): Pick $q \in \mathbb{Q}$. So $\pi(q) \in G$.
We show:

(2) $\pi[D \cap q\downarrow]$ is dense below $\pi(q)$

Given (2): Since $\pi(q) \in G$ and G is $(\mathbb{P}, V)$-generic,

$G \cap \pi[D \cap q\!\downarrow] \neq \emptyset$, i.e. there is some $d \in D$ s.t. $d \leq q$ and $\pi(d) \in G$. that is: $d \leq q$, $\underbrace{d \in \mathbb{Q} \text{ and } d \in D}_{d \in D \cap \mathbb{Q}}$

which proves (1).

$$\begin{tikzcd}
q \arrow[r, "\pi"] \arrow[d, dash] & \pi(q) \arrow[d, dash] \\
 & p \leq \pi(q) \arrow[d, dash] \\
q' \arrow[r] \arrow[d, dash] & \pi(q') \arrow[d, dash] \\
d \arrow[r] & \pi(d)
\end{tikzcd}$$

To see (2):

if $p \leq \pi(q)$ we can find some $q' \leq q$ s.t. $\pi(q') \leq p$. Since D is dense in $\mathbb{R}$, we can find some $d \in D$ s.t. $d \leq q'$. Then $\pi(d) \leq \pi(q') \leq p$. $\square$ (2)

We did not need that D is open here. $\square$ (a)

Proof of (b) Let $\mathbb{Q} = \mathbb{R}/(G, \pi)$. Obviously: $H \subseteq \mathbb{Q}$ is a filter. To see the genericity, pick an open dense set D of $\mathbb{Q}$ s.t. $D \in V[G]$. Using the Maximality principle we can fix $\mathbb{P}$-terms $\dot{\mathbb{Q}}, \dot{D}$ such that

(3) $\dot{\mathbb{Q}}^G = \mathbb{Q}$ $\quad \dot{D}^G = D$

(4) $\Vdash^V_{\mathbb{P}} \dot{\mathbb{Q}} = \check{\mathbb{R}}/(\dot{G}, \check{\pi})$, and

$\Vdash^V_{\mathbb{P}} \dot{D}$ is an open dense subset of $\dot{\mathbb{Q}}$

(Exercise.)

Now define

$$D' = \{ d \in \mathbb{R} \mid \pi(d) \Vdash^V_{\mathbb{P}} \check{d} \in \dot{D} \} \in V$$

We show:

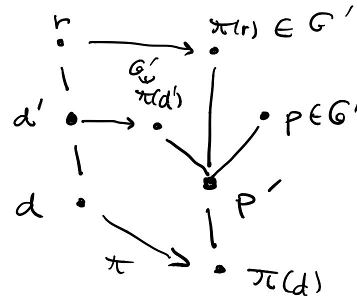
(5) D' is a dense subset of $\mathbb{R}$

Given (5): Since H is $(\mathbb{R}, V)$-generic, there is some $d \in H \cap D'$. Since $d \in H$: $\pi(d) \in G$. Since $d \in D'$:

$\pi(d) \Vdash^V_{\mathbb{P}} \check{d} \in \dot{D}$

hence $d \in \dot{D}^G = D$. Hence $d \in H \cap D$, i.e. $H \cap D \neq \emptyset$.
This then proves (b).

To see (5): Pick an element $r \in \mathbb{R}$. Consider any $(\mathbb{P}, V)$-generic G' s.t. $\pi(r) \in G'$.



Then $r \in \mathbb{R}/(G', \pi) = \dot{\mathbb{Q}}^{G'}$

Because $\dot{D}^{G'}$ is dense in $\dot{\mathbb{Q}}^{G'}$ we can find some $d' \in \dot{D}^{G'}$ such that $d' \leq r$. So by Forcing theorem (B) we can find some $p \in G'$ s.t. $p \Vdash^V_{\mathbb{P}} \check{d}' \in \dot{D}$.

Since $\pi(d'), \pi(r), p \in G'$, we can find some $p \in G'$ s.t. $p \leq \pi(d'), \pi(r), p$.

Finally: Since $p \leq \pi(d')$ and π is a projection, we can find some $d \leq d'$ s.t. $\pi(d) \leq p'$.

To summarize:

- $p \Vdash_{\mathbb{P}} \check{d}' \in \dot{D}$, so $\pi(d) \Vdash_{\mathbb{P}} \check{d}' \in \dot{D}$
- $\pi(d) \Vdash \check{d} \in \dot{\mathbb{Q}}$ (trivial)
- By the two above and the fact that $d \leq d'$ and $\Vdash_{\mathbb{P}} \dot{D}$ is open:

$$\pi(d) \Vdash \check{d} \in \dot{D}$$

So given $r \in \mathbb{R}$ we found such a d. □ (5) □ (6).

Forgot to check in (a): $G = \pi[H]\uparrow$. We do have: $\pi[H]\uparrow \subseteq G$. Now by P 13.22: $\pi[H]\uparrow$ is a $(\mathbb{P}, V)$-generic filter. So by maximality, $\pi[H]\uparrow = G$. □ T 13.25.

13.26. Example Product. Given posets $\mathbb{P}, \mathbb{Q}$ we define the product $\mathbb{R} = \mathbb{P} \times \mathbb{Q}$ as follows:

- Conditions ordered pairs $(p,q) \in \mathbb{P} \times \mathbb{Q}$
- Ordering $(p,q) \le (p',q')$ iff $(p \le p'$ and $q \le q')$

It is easy to check that the map

$$\pi_1 : \mathbb{P} \times \mathbb{Q} \longrightarrow \mathbb{P}$$

defined by

$$\pi_1(p,q) = p$$

is a projection.

(A) If H is $(\mathbb{P}\times\mathbb{Q}, \mathcal{V})$-generic, then by T 13.22 $\pi[H]\uparrow$ is $(\mathbb{P},\mathcal{V})$-generic. In our case $\pi[H]\uparrow = \pi[H]$

$$\pi[H] = \{ p \in \mathbb{P} \mid (\exists q \in \mathbb{Q})((p,q) \in H)\}$$

(B) Assume G is $(\mathbb{P},\mathcal{V})$-generic. Then the conditions in $\mathbb{P}\times\mathbb{Q}/(G,\pi_1)$ are pairs $(p,q) \in G \times \mathbb{Q}$ and the ordering is the ordering of the product.

This quotient is highly non-separative: if $p,p' \in G$ then $(p,q) \le^* (p',q)$.

We define a map

$$e : \mathbb{P}\times\mathbb{Q}/(G,\pi_1) \longrightarrow \mathbb{Q}$$

by

$$e(p,q) = q$$

Claim e is a dense embedding.

- order-preservation: Obvious
- incompatibility-preservation: If $e(p,q) \| e(p',q')$ i.e. $q \| q'$, fix some $q^* \in \mathbb{Q}$ s.t. $q^* \le q, q'$. Because $p,p' \in G$ we can find some $p^* \in G$ s.t. $p^* \le p,p'$. Then $(p^*,q^*) \in \mathbb{P}\times\mathbb{Q}/(G,\pi_1)$

and $(p^*, q^*) \leq (p,q)$, (p',q')

• Density: Easy, as e is surjective.

(C) Now if K is $(\mathbb{P}\times\mathbb{Q}, V)$-generic and $G = \pi_1[K]$ then K is $(\mathbb{P}\times\mathbb{Q}/(G,\pi_1), V[G])$-generic. Now if e is as in (B), let

$$H = e[K]\uparrow = e[K].$$

then H is $(\mathbb{Q}, V[G])$-generic, and

$$H = \{q \in \mathbb{Q} \mid (\exists p \in \mathbb{P})((p,q) \in K)\}$$
$$= \pi_2[K] \text{ where}$$

$$\pi_2 : \mathbb{P}\times\mathbb{Q} \longrightarrow \mathbb{Q} \quad \text{by} \quad \pi_2(p,q) = q.$$

One can check:

$$K = G \times H$$

<u>Summary</u> (i) If G is $(\mathbb{P}, V)$-generic and H is $(\mathbb{Q}, V[G])$-generic then

$$K = G\times H \text{ is } (\mathbb{P}\times\mathbb{Q}, V)\text{-generic}$$

(ii) if K is $(\mathbb{P}\times\mathbb{Q}, V)$-generic and we define G, H as above then $K = G\times H$ and G is $(\mathbb{P}, V)$-generic and H is $(\mathbb{Q}, V[G])$-generic.

(iii) Because $\times$ is symmetric, (i) + (ii) gives: if G is $(\mathbb{P}, V)$-generic and H is $(\mathbb{Q}, V[G])$-generic then H is $(\mathbb{Q}, V)$-generic and G is $(\mathbb{P}, V[H])$-generic

In this case G, H are called <u>Mutual Generics</u>

13.27. Example Composition of forcing posets.

Assume $\mathbb{P}$ is a poset and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-term s.t.

$$\Vdash^V_{\mathbb{P}} \dot{\mathbb{Q}} \text{ is a poset.}$$

We define the composition $\mathbb{P} * \dot{\mathbb{Q}}$ as follow:

- conditions: ordered pairs $(p, \dot{q})$ s.t. $p \in \mathbb{P}$ and $\Vdash^V_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$
- ordering: $(p, \dot{q}) \leq (p', \dot{q}')$ iff
  - $p \leq p'$
  - $p \Vdash^V_{\mathbb{P}} \dot{q} \leq_{\dot{\mathbb{Q}}} \dot{q}'$

One can do an analysis of $\mathbb{P} * \dot{\mathbb{Q}}$ similarly as for $\times$, but only for the first coordinate, as $*$ is not symmetric.

Forcing with $\mathbb{P} * \dot{\mathbb{Q}}$ is equivalent with forcing with $\mathbb{P}$ first over V and then with the evaluation of $\dot{\mathbb{Q}}$ over the corresponding generic extension via $\mathbb{P}$.

$$\pi : \mathbb{P} * \dot{\mathbb{Q}} \longrightarrow \mathbb{P}$$

$$\pi(p, \dot{q}) = p$$

is a projection.

Notice also that $\pi(1_{\mathbb{P} * \dot{\mathbb{Q}}}) = 1_{\mathbb{P}}$. The following theorem shows that the operations "composition" and "quotient" are "inverse" of each other

13.28. Theorem Let $\pi : \mathbb{R} \longrightarrow \mathbb{P}$ be a projection such that $\pi(1_{\mathbb{R}}) = 1_{\mathbb{P}}$. Then the map

$$e : \mathbb{R} \longrightarrow \mathbb{P} * \check{\mathbb{R}} / (\dot{G}, \check{\pi})$$

defined by

$$e(r) = (\pi(r), \dot{r})$$

is a dense embedding, where for each $r \in \mathbb{R}$, $\dot{r}$ is a $\mathbb{P}$-term satisfying

- $\Vdash_{\mathbb{P}}^{\vee} \dot{r} \in \check{\mathbb{R}} / (\dot{G}, \check{\pi})$

- $\pi(r) \Vdash \dot{r} = \check{r}$