13.20. Remark. Regarding strong master conditions from the discussion of Abraham-Shelah forcing: Recall: if \( \mathcal{P} \) is a point and \( p \in \mathcal{P} \) then

\[
\mathcal{P}_p = \{ p' \in \mathcal{P} \mid p' \leq p \} \quad \text{with the same ordering}
\]

( or \( \{ p' \in \mathcal{P} \mid p' \leq p \} \) )

One can check: \( p \) is a strong master condition for \( \mathcal{P} \cup X \) iff \( \odot (\mathcal{P}_p \cap X) \rightarrow \mathcal{P}_p \) is a regular embedding.

13.21. Definition. Assume \( \mathcal{P}, \mathcal{R} \) are posets. A map \( \pi : \mathcal{R} \rightarrow \mathcal{P} \) is a projection if the following hold:

(a) \( r \leq r' \Rightarrow \pi(r) \leq \pi(r') \)

(b) For every \( r \in \mathcal{R} \) and every \( p \in \pi(r) \)

there is an \( r' \leq r \) s.t. \( \pi(r') \leq p \)

In other words: \( \pi^{-1}(p) \) is dense below \( \pi(r) \).

13.22. Prop. Assume \( \pi : \mathcal{R} \rightarrow \mathcal{P} \) is a projection.

(a) If \( D \subseteq \mathcal{R} \) is open dense then \( \pi^{-1}(D) \) is dense in \( \mathcal{R} \)

(b) If \( H \subseteq (\mathcal{R}, \mathcal{V}) \) generic then \( G = \pi[H] \subseteq (\mathcal{P}, \mathcal{V}) \) generic

Proof: (b) is a direct consequence of (a). We prove (a). Pick \( r \in \mathcal{R} \)

\[
\pi(r) \quad \text{since} \quad D \text{ is dense in } \mathcal{P} \quad \text{we have some}
\]

\[
d' \in \mathcal{D} \quad \text{s.t.} \quad d' \leq \pi(r). \quad \text{Since } \pi \text{ is a projection}
\]

there is some \( d \leq r \) s.t. \( \pi(d) \leq d' \).

Because \( D \) is open: \( \pi(d) \in D \). So \( d \leq r \) and \( d \in \pi^{-1}(D) \). \( \square \)
13.23: Definition \( \pi \colon R \rightarrow P \) be a projection and \( G \) be \((P, V)^{-}\)generic. We define the quotient of \( R \) by \( G \) with respect to \( \pi \), denoted by \( R/(G, \pi) \), as follows:

- **Conditions** elements of \( \pi^{-1}[G] \)
- **Ordering** the restriction of \( \leq_R \) to \( \pi^{-1}[G] \).

Similarly, we can define quotient on lens of regular embeddings:

13.24: Definition \( \pi : R \rightarrow R \) is a regular embedding and \( G \) is \((P, V)^{-}\)generic then we define the quotient of \( R \) by \( G \) with respect to \( \pi \), denoted by \( R/(G, \pi) \) as follows:

- **Conditions** are all conditions \( r \in R \) s.t. \( \forall 1 \pi(p) \) for all \( p \in G \)
- **Ordering** is the restriction of \( \leq_R \) to the set of conditions in the quotient.

13.25: Theorem Assume \( \pi : R \rightarrow P \) is a projection.

(a) If \( G \) is \((P, V)^{-}\)generic and \( H \) is \((R, V(C))^{-}\)generic, where \( Q = R/(G, \pi) \) then \( H \) is \((R, V)^{-}\)generic and \( G = \pi^{-1}[H] \).

(b) If \( H \) is \((R, V)^{-}\)generic and \( G = \pi^{-1}[H] \), then

\( H \) is \((R/(G, \pi), V(G)^{-})\) generic.

**Proof** (a) Obvious: \( H \leq_R G \) is a filter. To see genericity:

let \( D \cap E \) be (open) dense in \( R \). We show:

1. \( D \cap Q \) is a dense subset of \( Q \).

Given (1): Since \( H \) is \((R, V(C))^{-}\)generic and \( D \cap Q \subseteq V(C) \)

we have \( \phi \neq H \cap (D \cap Q) \subseteq H \cap D \).

To see (1): Pick \( q \in Q \). So \( \pi(q) \in G \).

We show:

2. \( \pi^{-1}(D \cap q) \) is dense below \( \pi(q) \)

Given (2): Since \( \pi(q) \subseteq G \) and \( G \) is \((R, V)^{-}\)generic,
Given (5): Since \( H = (D, \pi) \) - generic, there is some \( d \in H \cap D' \). Since \( d \in H \cap D' \), \( \pi(d) \notin \pi(G) \). Since \( \pi(d) \notin \pi(G) \), we have:

\[ \pi(d) \notin \pi(G) \]

which proves (1).

To see (2):

\[ \text{For } p \in \pi(G), \text{ we can find some } q \in \pi(G) \text{ such that } \pi(q) \leq p. \text{ Since } D \text{ is dense in } \mathbb{R}, \text{ we can find some } d \in D \text{ such that } d \leq q'. \text{ Thus } \pi(d) \leq \pi(q') \leq p. \quad \square \text{(2)} \]

We did not need that \( D \) is open here. \( \Box \text{(2)} \)

Proof of (3): Let \( \mathcal{Q} = \mathbb{R}/(G, \pi) \). Obviously: \( H \subseteq \mathcal{Q} \) is a filter. To see the genericity, pick an open dense set \( D \) of \( \mathcal{Q} \) such that \( D \subseteq V(G) \). Using the Maximality principle, we can fix \( \mathbb{R} \)-terms \( \mathcal{Q}, D \) such that:

1. \( \mathcal{Q}^D = \mathcal{Q} \)
2. \( D^D = D \)
3. \( \mathcal{Q} = \mathbb{R}/(G, \pi) \)
4. \( \mathcal{Q} = \mathbb{R}/(G, \pi) \)

(Exercise)

Now define:

\[ D' = \{ d \in \mathbb{R} \mid \pi(d) \not\in \pi(G) \} \]

We show:

5. \( D' \) is a dense subset of \( \mathbb{R} \)

Given (5): Since \( H = (D, \pi) \) - generic, there is some \( d \in H \cap D' \). Since \( d \in H \cap D' \), \( \pi(d) \notin \pi(G) \). Since \( \pi(d) \notin \pi(G) \):

\[ \pi(d) \notin \pi(G) \]
hence $d \in D' \implies D' \cap H \cap D = \emptyset$.

This then proves (b).

To see (5): Pick any element $v \in R$. Consider any $(\pi, \pi')$-generic $G'$ so that $\pi(v) \in G'$.

Then $v \in \text{IIR}(G', \pi) = \varphi^G$. Because $D'$ is dense in $\varphi^G$, we can find some $d' \in D'$ such that $d' \in G'$.

By forcing theorem (3), we can find some $p \in G'$ so that $\pi(v) \Vdash d' \in D$.

Since $\pi(d'), \pi(v), p \in G'$, we can find some $p' \in G'$ so that $p \subseteq \pi(d'), \pi(v), p'$.

Finally, since $p \subseteq \pi(d')$ and $\pi$ is a projection, we can find some $d \in d'$ so that $\pi(d) \subseteq p'$.

To summarize:

- $p \Vdash d' \in D$, so $\pi(d) \Vdash d' \in D'$
- $\pi(d) \Vdash d' \in G'$ (trivial)

By the two above and the fact that $d \in d'$ and $\pi(d) \Vdash p$ it is open:

$\pi(d) \Vdash d \in D$.

So given $v \in R$ we find such a $d$. \( \Box \) (5) \( \Box \) (6).

Forgot to check in (c1): $G = \pi(C) \uparrow$. We do have: $\pi(C) \uparrow \subseteq G$.

Now by P 13.22: $\pi(C) \uparrow$ is a $(\mathcal{P}, \mathcal{N})$-generic filter. So by maximality, $\pi(C) \uparrow = G$. \( \Box \) (13.25).
13.26 Example Product. Given points \( P, Q \) we define the product \( R = P \times Q \) as follows:

- **Condition** ordered pairs \((p, q) \in P \times Q\)
- **Ordering** \((p, q) \preceq (p', q')\), if \((p \leq p'\) and \(q \leq q')\)

It is easy to check that the map

\[ \pi_1 : P \times Q \to P \]

defined by

\[ \pi_1(p, q) = p \]

is a projection.

(A) If \( H \) is \((P \times Q, \preceq)\) - generic, then by \( T \) (3.22)

\[ \pi[H] \subseteq (P, \preceq)\) - generic. In our case \( \pi[H] = \pi[H] \]

\[ \pi[H] = \{ p \in P | \exists q \in Q (p, q) \in H \} \]

(B) Assume \( G \) is \((P, \preceq)\) - generic. Then the condition

in \( P \times Q \setminus (G, \pi_1) \) are pairs \((p, q) \in G \times Q\)

and the ordering is the ordering of the product.

This quotient is highly non-separative: if \( p, p' \in G\)

then \((p, q) \preceq (p', q')\).

We define a map

\[ \iota : P \times Q / (G, \pi_1) \to Q \]

by

\[ \iota(p, q) = q \]

**Claim**: \( \iota \) is a dense embedding.

- **order - preservation**: obvious
- **incompatibility - preservation**: \( \iota \) \( (p, q) \preceq (p', q') \)

\( \Rightarrow q \preceq q' \), \( \exists q^* \in Q \) s.t. \( q^* \leq q, q' \).

Because \( p, p' \in G \) we can find some \( p^* \in G \) s.t.

\( p^* \leq p, p' \). Then \((p^*, q^*) \in P \times Q \setminus (G, \pi_1) \).
and $(p^*, q^*) \in (p, q), (p', q')$

- **Domain:** Easy, as $e$ is surjective.

**Case (c)** Now if $K \subseteq (P \times Q, \preceq)$ - generic and

$G = \pi_1[K]$ then $K \subseteq (P \times Q, (\pi_1, \sim), \preceq[G])$ -

- generic. Now if $e$ is as in (B), let

$H = e[K] \uparrow = e[K]$,

then $H \subseteq (Q, \preceq[e])$ - generic, and

$H = \{ q \in Q \mid \exists p \in P \exists \pi_2[G](\langle p, q \rangle \in K) \}$

$\pi_2[K]$ where

$\pi_2 : P \times Q \to Q$ by $\pi_2(p, q) = q$.

One can check:

$K = G \times H$

**Summary:**

(i) If $G \subseteq (P, \preceq)$ - generic and

$H \subseteq (Q, \preceq[G])$ - generic then

$K = G \times H \subseteq (P \times Q, \preceq)$ - generic

(ii) If $K \subseteq (P \times Q, \preceq)$ - generic and we define

$G, H$ as above then $K = G \times H$ and

$G \subseteq (P, \preceq)$ - generic and $H \subseteq (Q, \preceq[G])$ - generic.

(iii) Because $x$ is symmetric, (c) + (ii) give:

If $G$ is $(P, \preceq)$ - generic and $H$ is $(Q, \preceq[G])$ - generic

then

$H$ is $(Q, \preceq)$ - generic and $G$ is $(P, \preceq[H])$ - generic.

In this case $G \cap H$ are called **Mutual Generic**
Example

Composition of forcing posets.

Assume $P$ is a poset and $Q$ is a $P$-dense set.

We define the composition $P \times Q$ as follows:

- **Conditions**: ordered pairs $(p, q)$ s.t. $p \in P$ and $\forall q' \in Q.

- **Ordering**: $(p, q) \leq (p', q')$ if

\[- p \leq p',
\]
\[- q \leq q'.
\]

One can do an analysis of $P \times Q$ similarly as for $x$, but only for the first coordinate.

Forcing with $P \times Q$ is equivalent with forcing with $P$ first over $V$ and then with the evaluation of $Q$ over the corresponding generic extension via $P$.

$$
\pi : P \times Q \rightarrow P
$$

$$
\pi(p, q) = p
$$

is a projection.

Notice also that $\pi(1_{P \times Q}) = 1_P$. The following theorem shows that the operations "composition" and "quotient" are "inverse" of each other.
Theorem

Let \( \pi : R \to P \) be a projection such that \( \pi(1_P) = 1_P \). Then the map

\[ \epsilon : R \to P \ast \tilde{R} / (G, \pi) \]

defined by

\[ \epsilon(r) = (\pi(r), \tilde{r}) \]

is a dense embedding, where for each \( r \in R \),

\( \tilde{r} \) is a \( P \)-term satisfying

- \( \forall \tilde{r} \in \tilde{R} / (G, \pi) \)
- \( \pi(r) \ast \tilde{r} = \tilde{r} \)