The Mostowski Collapsing Theorem

A relation R on a set A is *extensional* iff for every $x, y \in A$ we have:

$$[(\forall z \in A)(zRx \leftrightarrow zRy)] \longleftrightarrow x = y. \tag{1}$$

Use the construction by transfinite recursion to prove the so-called Mostowski Collapsing Theorem:

Theorem 1 *If R is a binary relation on a set A and R is both extensional and well-founded, then there is a unique transitive set U and a unique isomorphism $\sigma : \langle A, R\rangle \longrightarrow \langle U, \in\rangle$.*

Conclude that if U_1, U_2 are transitive sets and the structures $\langle U_1, \in\rangle$ and $\langle U_2, \in\rangle$ are isomorphic, then $U_1 = U_2$ and the isomorphism in question is the identity map.

Hint. The isomophism is defined by recursion in the only possible way, that is, as follows:

$$\sigma(x) = \{\sigma(z);\ zRx\}.$$

Here is the obvious version of the Mostowski Collapsing Theorem for proper classes:

Theorem 2 *If R is a set-like binary relation on a class A and R is both extensional and well-founded, then there is a unique transitive class U and a unique class isomorphism $\sigma : \langle A, R\rangle \longrightarrow \langle U, \in\rangle$.*

Consequently, if U_1, U_2 are two transitive classes and $\sigma : \langle U_1, \in\rangle \to \langle U_2, \in\rangle$ is a class isomorphism then $U_1 = U_2$ and $\sigma = \text{id}$.