WEEK 6

1. Let $\langle X, \prec \rangle$ be a linear ordering that satisfies the c.c.c., but is not separable. Show that there is a linear ordering $\langle Y, \prec \rangle$ that

   (a) satisfies the c.c.c., but is not separable,

   (b) is dense and without endpoints,

   (c) is Dedekind complete.

   **Hint.** The point is to arrange (b), as (c) can be arranged by forming a Dedekind completion and verifying that it preserves (a) and (b). Given $x, y \in X$, let $x \sim y$ just in case that the interval $(\min\{x, y\}, \max\{x, y\})$ contains a countable dense subset. Notice that $\sim$ is an equivalence relation on $X$; let $Y' = X/\sim$ and $\prec$ be the linear ordering on $Y'$ induced by $\sim$. Show that $\langle Y', \prec \rangle$ is a dense linear ordering that satisfies (a). The key point is the observation that each equivalence class is separable; the proof of this observation builds on the fact that $\langle X, \prec \rangle$ satisfies the c.c.c. $Y'$ can have endpoints, but if we remove them, we obtain a linear ordering satisfying both (a) and (b). This is our desired $\langle Y, \prec \rangle$.

2. Show that $\Diamond$ implies the Continuum Hypothesis.

   **Hint.** Show that every subset of $\omega$ is on a $\Diamond$-sequence.

3. Given a cardinal $\kappa$ and a set $X$ of cardinality $\kappa$, we say that two sets $A, B \subseteq X$ are *almost disjoint* just in case that $A \cap B$ is of size smaller than $\kappa$. We say that $A \subseteq \mathcal{P}(X)$ is an *almost disjoint family* just in case that $A$ consists of pairwise almost disjoint sets of cardinality $\kappa$.

   (a) Show that there is an almost disjoint family $A \subseteq \mathcal{P}(\omega)$ such that $\text{card}(A) = 2^\omega$. Notice that any disjoint family of subsets of $\omega$ is of size at most $\omega$.

   (b) Let $\kappa$ be regular and let $\{x_\xi; \xi < \kappa\}$ be an almost disjoint family of subsets of $\kappa$. Modify the Cantor diagonal argument to construct a set $x \subseteq \kappa$ that is almost disjoint with each $x_\xi$.

   (c) Use (b) to construct, by transfinite recursion on $\kappa^+$, an almost disjoint family $A \subseteq \mathcal{P}(\kappa)$ of cardinality $\kappa^+$. In general, this is the largest almost disjoint family that can be constructed in ZFC alone.
(d) Assume that $\diamondsuit_\kappa$ holds. Show that there is an almost disjoint family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ of size $2^\kappa$ which consists of stationary sets (stationary in $\kappa$, of course).

**Hint.** (a) Find an almost disjoint family $\mathcal{A}' \subseteq \mathcal{P}(^{<\omega}\{0,1\})$ of size $2^\omega$. Recall that there are $2^\omega$ functions $G : \omega \to \{0,1\}$.

(b) If $\xi < \kappa$, we can fix some $\gamma < \kappa$ such that $x_\xi - \gamma$ is disjoint with $x_{\bar{\xi}} - \gamma$ for all $\bar{\xi} < \xi$.

(c) At each step of the recursion, enumerate the family constructed so far in the order type $\kappa$.

(d) Fix some $\diamondsuit_\kappa$-sequence $\langle A_\alpha ; \alpha < \kappa \rangle$. Given any $A \subseteq \kappa$, consider the set $S_A = \{ \xi < \kappa ; A \cap \xi = A_\xi \}$. 