WEEK 8

1. An automorphism of a tree $T$ is a bijective map $\sigma : T \to T$ such that for every $s, t \in T$ we have

$$s <_T t \iff \sigma(s) <_T \sigma(t).$$

A tree $T$ is rigid just in case that there are no nontrivial automorphisms of $T$. (That is, the only automorphism of $T$ is the identity.)

Assume $\Diamond (S)$ holds where $S \subseteq \omega_1$. Construct a rigid Suslin tree.

**Hint.** This is like the construction of a Suslin tree from the lecture with the only difference: You have to use $\Diamond (S)$ to kill not only all antichains, but also all automorphisms of $T$.

This is a more demanding exercise. Ask me if you need more hints.

2. Given a regular cardinal $\kappa$, a $\Diamond _\kappa ^#$-sequence is a sequence of the form $\langle A_\alpha ; \alpha < \kappa \rangle$ where each $A_\alpha \subseteq \alpha$ with the following property:

- For every $A \subseteq \kappa$ there is a closed unbounded set $C \subseteq \kappa$ such that for every $\alpha \in C$ we have $A \cap \alpha = A_\alpha$.

The principle $\Diamond _\kappa ^#$ postulates the existence of a $\Diamond _\kappa ^#$-sequence.

Show that the principle $\Diamond _\kappa ^#$ is false, i.e. there is no $\Diamond _\kappa ^#$-sequence.

3. In the lecture we formulated the principle $\Diamond _\kappa$ for regular $\kappa$, but the same formulation makes sense also for singular $\kappa$. Show that $\Diamond _\kappa$ fails for every singular $\kappa$.

**Hint.** Given any potential $\Diamond _\kappa$-sequence $\{ A_\alpha \}_\alpha$, use the fact that you have a closed unbounded set of ordertype $\mu < \kappa$ to construct (by a diagonal argument) a set $A \subseteq \mu$ that is not captured by $\{ A_\alpha \}_\alpha$.