WEEK 10

1. Let \( P \) be the poset defined as follows:
   
   - **Conditions** are finite functions \( p : a \to \omega_1 \) where \( a \subseteq \omega \);

   - **Ordering** is “extension”, i.e. \( p \leq q \) just in case that \( p \supseteq q \).

   Prove:

   (a) \( P \) fails to satisfy the c.c.c.

   (b) There is a family \( D = \{ D_\alpha ; \alpha < \omega_1 \} \) of dense subsets of \( P \) such that no filter on \( P \) is \( D \)-generic.

   This shows that \( \text{MA}(\omega_1) \) cannot be extended to posets that fail to satisfy the c.c.c.

2. Show that \( \text{MA}(\kappa) \) is equivalent to the following statement:

   If \( P \) is a c.c.c. poset of size at most \( \kappa \) and \( D \) is a family of dense sets in \( P \) with \( |D| \leq \kappa \) then there is a \( D \)-generic filter.

   Thus, to verify \( \text{MA}(\kappa) \), it suffices to check posets of size at most \( \kappa \).

   **Hint.** For the nontrivial direction, let \( P \) be an arbitrary poset and let \( \{ D_\xi ; \xi < \kappa \} \) be a family of dense sets in \( P \). For each \( \xi < \kappa \), let \( \check{D}_\xi \) be a predicate symbol and let \( L \) be the first-order language which has the symbol \( \leq \) and all symbols \( \check{D}_\xi \). View \( \langle P, \{ D_\xi ; \xi < \kappa \} \rangle \) as a first order structure for \( L \) where \( \leq \) is interpreted as the partial ordering \( \leq \) of \( P \) and \( \check{D}_\xi \) as \( \check{D}_\xi \). Apply the downward Löwenheim-Skolem theorem.

3. Prove

   \[
   \text{MA}(\omega_1) \Rightarrow \text{SH}.
   \]

   Recall that \( \text{SH} \) is the Suslin Hypothesis, i.e. it asserts that there is no Suslin tree.

   **Hint.** Assume there is a Suslin tree, say \( T \). Turn \( T \) upside down.