WEEK 5

1. (ZFC) Prove the Ramsey theorem which asserts that $\omega \rightarrow (\omega)^n_m$ for all integers $m, n > 0$. Use the tree argument.

   **Hint.** Proceed by induction on $n$. Regarding the induction step from $n$ to $n + 1$: Try to combine ideas from the proof of the property $\omega \rightarrow (\omega)^2_2$ and those form the proof of the property $\kappa \rightarrow (\kappa)^n_\kappa$ for weakly compact $\kappa$. What you have to do is to construct a finitely branching tree $T$ on $\omega$ (in the lecture we constructed a subtree of $\langle \omega, 2 \rangle$ labeled by integers, but in the general case it might be simpler to construct a tree directly on $\omega$) of height $\omega$ such that for every branch $b$ through $T$, every $a \in [b]^n$ and every $i, j \in b$ that are above all elements of $a$ in the sense of $<_T$, the sets $a \cup \{i\}$ and $a \cup \{j\}$ have the same colour.

2. (ZFC) Assume there is a countably complete uniform ultrafilter on some set $I$. Show that there is a measurable cardinal.

   **Hint.** Argument 1: Use the ultrapower construction. Argument 2: Observe that $I$ can be taken to be a cardinal. Let $\kappa$ be the least such cardinal. Show that any $\sigma$-complete uniform ultrafilter on $\kappa$ is actually $\kappa$-complete: If the completeness of such an ultrafilter were $\gamma^+$ for some $\gamma < \kappa$, there would be a countably complete uniform ultrafilter on $\gamma$. 

1