WEEK 8

1. Let $\mathbb{P}$ be a poset and $\sigma, \sigma'$ be two $\mathbb{P}$-names. Find a $\mathbb{P}$-name $\tau$ such that for every $\mathbb{P}$-generic filter $G$ the following holds:

   (a) $\tau_G = \{\sigma_G, \sigma'_G\}$;
   
   (b) $\tau_G = \langle \sigma_G, \sigma'_G \rangle$;
   
   (c) $\tau_G = \bigcup \sigma_G$.

   **Hint.** (c) Notice that if $\rho_G \in \bigcup \sigma_G$ then there are $p, q \in \mathbb{P}$ and a $\mathbb{P}$-name $\rho'$ such that $\langle p, \rho' \rangle \in \sigma$, $\langle q, \rho \rangle \in \rho'$ and $p, q$ are compatible.

2. Assume $\mathbb{P}$ is a poset in $M$ and $A, B \in M$. Let $p \in \mathbb{P}$ and $\dot{f}$ be a $\mathbb{P}$-name such that

   $$p \Vdash \dot{f} : \dot{A} \rightarrow \dot{B}.$$ 

   Let $q \leq p$, $a \in A$ and $b \in B$ be such that

   $$q \Vdash \dot{f}(\dot{a}) = \dot{b}.$$ 

   Show that if $c \in B$ is such that $c \neq b$ and $q' \in \mathbb{P}$ is compatible with $q$ then

   $$q' \not\Vdash \dot{f}(\dot{a}) = \dot{c}.$$ 

3. Let $\mathbb{P}$ be a separative poset in $M$ and let $G$ be a $\mathbb{P}$-generic filter over $M$. Let $F \in M$ be such that $F \subseteq G$. Show that there is some $p \in \mathbb{P}$ such that

   $$p \leq r \text{ for all } r \in F.$$ 

   **Hint.** This is a density argument. Try to find a dense set $D$ in $\mathbb{P}$ such that any element $p \in D \cap G$ satisfies the above condition. (See the arguments we did in class for inspiration.)

4. Generalize the argument we had when we constructed canonical names for subsets of $A$ (this came in computing of the upper bound on the size of $\mathcal{P}(\omega)$ in $M[G]$).
Thus, \( \mathbb{P} \) is a poset in \( M \) and we have \( \mathbb{P} \)-names \( \sigma, \tau \) and some \( p \in \mathbb{P} \) such that
\[
p \models \sigma \subseteq \tau.
\]
Find a \( \mathbb{P} \)-name \( \sigma' \) such that \( \text{rng}(\sigma') \subseteq \text{rng}(\tau) \) (that is, if \( \langle r, \rho \rangle \in \sigma' \) then \( \langle r', \rho \rangle \in \tau \) for some \( r' \)) and
\[
p \models \sigma = \sigma'.
\]