WEEK 9

1. Let $M$ be an arbitrary ground model. In $M$, consider the poset $\text{col}(\omega_1, 2^{\omega})$. Let $G$ be a generic filter for this poset over $M$. Show:

(a) $M$ and $M[G]$ have the same subsets of $\omega$.

(b) In $M[G]$, there is a surjection $g : \omega_1 \to \mathcal{P}(\omega)$, i.e. $M[G] \models \text{CH}$.

(c) In $M$, the poset $\text{col}(\omega_1, 2^{\omega})$ satisfies the $(2^{\omega})^+\text{-c.c.}$

(d) $\omega_1^{M[G]} = \omega_1^M$, $\omega_2^{M[G]} = ((2^{\omega})^+)^M$ and all cardinals and cofinalities $> 2^{\omega}$ are preserved.

**Hint.** (c) use the $\Delta$-system lemma in the usual way. Notice that you do NOT have GCH in $M$, so you have to use general facts from cardinal arithmetic only.

2. Let $M \models \text{GCH}$. Let $\kappa$ be regular in $M$ and $\lambda$ be a cardinal in $M$ such that $\text{cf}^M(\lambda) > \kappa$. Force with $\text{Add}(\kappa, \lambda)$ over $M$. Show that if $G$ is a generic filter for this poset over $M$, then

(a) All cardinals and cofinalities are preserved.

(b) $M[G] \models 2^\kappa = \lambda$.

**Hint.** Follow the computation we did in lecture with $\text{Add}(\omega, \lambda)$. This time you have to use the chain condition for $\text{Add}(\kappa, \lambda)$ from the lecture, as well as the fact that your forcing is $\kappa^+$-closed.

3. Start with a model $M$ that satisfies the GCH. In $M$, let $\kappa < \kappa'$ be regular cardinals and $\lambda \leq \lambda'$ be cardinals such that $\text{cf}(\lambda) > \kappa$ and $\text{cf}(\lambda') > \kappa'$. Force first with $\text{Add}(\kappa', \lambda')$ over $M$, getting a generic filter $G$ and then with $\text{Add}(\kappa, \lambda)$ over $M[G]$, getting a generic filter $H$. Show:

$$M[G][H] \models 2^\kappa = \lambda \ & 2^{\kappa'} = \lambda'.$$

**Hint.** You can refer to Problem 2 above. The only tricky part is computing the upper bound for $2^\kappa$. Follow the usual steps. However, this time you don’t have GCH in $M[G]$. To compute the size of the set of all canonical names for subsets of $\kappa$, use the fact that $\text{Add}(\kappa', \lambda')$ is $\kappa'$-closed.
4. Let $M$ be arbitrary, let $\kappa$ be regular in $M$ and $\lambda > \kappa$ be strongly inaccessible in $M$. Force with $\text{Col}(\kappa, < \lambda)$ over $M$, getting a generic filter $G$. Show:

(a) $\text{Col}(\kappa, < \lambda)$ satisfies the $\lambda$-c.c.

(b) All cardinals and cofinalities $\leq \kappa$ and $\geq \lambda$ are preserved.

(c) $\lambda = \kappa^+ M[G]$. 