WEEK 1

1. Given two sequences $f, g \in \omega^\omega$, we write $f <^* g$ just in case that $g$ eventually dominates $f$, precisely:

$$f <^* g \iff f(n) < g(n) \text{ for all but finitely many } n \in \omega.$$ 

One can easily check that $<^*$ is a partial ordering on $\omega^\omega$. An easy diagonal argument yields that every countable $\mathcal{F} \subseteq \omega^\omega$ has an upper bound with respect to $<^*$, i.e. there is some $f \in \omega^\omega$ such that $\tilde{f} <^* f$ for all $\tilde{f} \in \mathcal{F}$. (This is a good warming-up exercise!)

Prove: MA($\kappa$) implies that every $\mathcal{F} \subseteq \omega^\omega$ of size at most $\kappa$ has an upper bound. Precisely: There is some $f \in \omega^\omega$ such that $\tilde{f} <^* f$ for all $\tilde{f} \in \mathcal{F}$.

Conclude that under MA, every maximal $<^*$-chain has size continuum.

**Hint.** Design a poset $\mathbb{P}$ that comprises all approximations to your upper bound $f$. Use a similar idea as that for almost disjoint forcing. Your conditions will be pairs of the form $\langle s, F \rangle$ where $s : u \to \omega$ is finite with $u \subseteq \omega$, and $F \subseteq \mathcal{F}$ is finite as well. The order $\leq$ on $\mathbb{P}$ is defined as follows:

$$\langle s', F' \rangle \leq \langle s, F \rangle$$

just in case that

- $s' \supseteq s$;
- $F' \supseteq F$;
- If $n \in \text{dom}(s') - \text{dom}(s)$ then $s'(n) > f(n)$ for all $f \in F$.

Define the generic sequence $g : \omega \to \omega$. The last of the three above requirements will guarantee that your generic sequence $g$ will majorize each $f \in F$ at all $n$ that are outside $s$, and therefore at all but finitely many $n$. Indeed, whenever you extend $s$, you have to do it in such a manner that you majorize all $f \in F$.

You have to show:

- $\mathbb{P}$ satisfies the c.c.c.
- $g$ is really a function defined on the entire $\omega$.
- If $f \in \mathcal{F}$ then $f(n) < g(n)$ for all but finitely many $n$. 

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The latter two assertions follow by a density argument.

2. Let $\mathbb{B}$ be a Boolean algebra. We let $X(\mathbb{B})$ be the set of all ultrafilters on $\mathbb{B}$. To each $b \in \mathbb{B}$, let

$$B_b = \{ U \in X(\mathbb{B}) ; b \in U \}.$$ 

Finally let

$$\mathcal{B} = \{ B_b ; b \in \mathbb{B} \}.$$ 

Then $\mathcal{B} \subseteq \mathcal{P}(X(\mathbb{B}))$, and together with the operations $\cap, \cup, \varnothing, X(\mathbb{B})$ and $-$ can be viewed as a subalgebra of the Boolean algebra $\mathcal{P}(X(\mathbb{B}))$.

Show: The assignment $h : \mathbb{B} \to \mathcal{B}$ defined by $h(b) = B_b$ is an isomorphism between the Boolean algebras $\mathbb{B}$ and $\langle \mathcal{B}, \cap, \cup, \varnothing, X(\mathbb{B}), - \rangle$. Thus, every Boolean algebra can be represented as an algebra of sets, i.e. the elements of the algebra can be represented as subsets of the set $X(\mathbb{B})$, the operations $\wedge$ and $\vee$ can be represented as the intersection and union, $0$ as the empty set, $1$ as the full set, and the complementation as the usual set complementation.

The representation just described is called the **Stone representation** of the Boolean algebra $\mathbb{B}$. 