WEEK 2

1. Let $X$ be a topological space with a countable base. In the following steps show that, under $\text{MA}(\kappa)$, the meager ideal is $\kappa^+$-complete. The arguments here resemble very much those which are used in the analysis of almost disjoint forcing.

Fix a countable base of topology $\mathcal{B}$. You have to show that the union of $\leq \kappa$ many meager sets is meager.

(a) Observe that it suffices to show: If $\mathcal{C}$ is a family of closed nowhere dense sets with $\text{card}(\mathcal{C}) \leq \kappa$ then $\bigcup \mathcal{C}$ is meager.

(b) Define a poset $\mathbb{P}$ as follows:

**Conditions** are pairs of the form $\langle p, F \rangle$ where
- $p$ is a function whose domain is a finite subset of $\omega$ and for each $n \in \text{dom}(p)$, the value $p(n)$ is a finite subset of $\mathcal{B}$;
- $F$ is a finite subset of $\mathcal{C}$.

**Ordering** is defined as follows

$$\langle p', F' \rangle \leq \langle p, F \rangle$$

just in case that
- $\text{dom}(p') \supseteq \text{dom}(p)$;
- $p'(n) \supseteq p(n)$ whenever $n \in \text{dom}(p)$;
- $F' \supseteq F$;
- if $n \in \text{dom}(p') - \text{dom}(p)$ then $\bigcup p'(n)$ is disjoint with every $C \in F$.

(c) The idea of the definition of $\mathbb{P}$ is the following. The sequences $p$ are approximations to a countable sequence of closed nowhere dense sets. If we choose a sufficiently generic $G$ on $\mathbb{P}$, let

$$g = \bigcup \{p; \langle p, F \rangle \in G \text{ for some } F\}.$$ 

For each $n \in \omega$, let $A_n = \bigcup g(n)$. Then each $A_n$ will be an open dense set in $X$, so $B_n \overset{\text{def}}{=} X - A_n$ will be closed nowhere dense. Moreover, $\bigcup \mathcal{C} \subseteq \bigcup \{B_n; n \in \omega\}$. Thus, $g$ codes a countable sequence of closed nowhere dense sets that covers $\bigcup \mathcal{C}$.
(d) For each \( n \in \omega \) find a countable family of dense sets \( D_n \) in \( \mathbb{P} \) such that if \( G \) is \( D_n \)-generic then the corresponding \( g \) satisfies that \( \bigcup g(n) \) is an open dense set in \( X \).

(e) For each \( C \in \mathcal{C} \) find a dense set \( D_C \) in \( \mathbb{P} \) such that if \( G \) meets \( D_C \) and is sufficiently generic, then \( X - \bigcup g(n) \) covers \( C \) whenever \( n \) is suitably chosen.

(f) Put (d) and (e) together and find a family \( \mathcal{D} \) of dense sets in \( \mathbb{P} \) such that if \( G \) is \( \mathcal{D} \)-generic, then we obtain \( g \) as in (c).

(g) Show that \( \mathbb{P} \) satisfies the c.c.c. so that you can apply MA(\( \kappa \)).

2. Return to the Stone representation of Boolean algebras from the Week 1 assignment. Let \( \mathbb{B} \) be a Boolean algebra.

(a) Show that the family \( \mathcal{B} = \{ B_b; \ b \in \mathbb{B} \} \) is a base of some topology \( \mathcal{T} \) on the set of all ultrafilters \( X(\mathbb{B}) \). The topological space \( \langle X(\mathbb{B}), \mathcal{T} \rangle \) is called the Stone space of \( \mathbb{B} \), and \( \mathcal{T} \) is called the Stone topology.

(b) Show that the Stone topology is Hausdorff.

(c) Show that each set \( B_b \) is clopen, i.e. it is both open and closed in the Stone topology.

(d) Show that the Stone space is compact.

**Hint.** First show that \( \mathcal{B} \) is a base for the collection of all closed sets, i.e. each closed set can be expressed as the intersection of some \( \mathcal{F} \subseteq \mathcal{B} \). Then observe that showing that \( X(\mathbb{B}) \) is compact boils down to showing that if \( \mathcal{F} \subseteq \mathcal{B} \) is a family with the finite intersection property then \( \bigcap \mathcal{F} \neq \emptyset \). Finally show that \( \bigcap \mathcal{F} \neq \emptyset \) for any such \( \mathcal{F} \); here you use the prime ideal theorem.