HOMEWORK WEEK 3

Remark. Problems marked with ($\star$) or more stars are more challenging. The more stars the more challenge.

1. Let $\langle X, <_X\rangle$ and $\langle Y, <_Y\rangle$ be two countable dense linear orderings without endpoints. Prove that they are isomorphic, that is, show the existence of a bijection $f : X \to Y$ such that $x <_X x' \iff f(x) <_Y f(x')$ for all $x, x' \in X$.

Hint. Fix enumerations $X = \{x_n \mid n \in \omega\}$ and $Y = \{y_n \mid n \in \omega\}$. Then construct an isomorphism f by induction on n using the back-and-forth construction: $f(x_0)$ can be defined arbtrarily. Then define the inverse image $f^{-1}(y_0)$ (if $f(x_0) \neq y_0$ so that the finite approximation to the map obtained this way is injective and order-preserving (in both directions). Inductively, granting that $f(0), \ldots, f(n)$ and $f^{-1}(y_0), \ldots, f^{-1}(y_n)$ have been defined, extend the definition of f to x_{n+1} in the domain and y_{n+1} in the range (if not yet defined) in a way that the injectivity and preservation properties of f are kept. Show that after ω steps you obtain an isomorphism f.

Also, think why the back-and-forth approach is used here.

2. Given a linear ordering $\langle L, <_L\rangle$ and a set $X \subseteq L$, we say that an element $x \in L$ is the supremum of X and write $x = \sup(X)$ if and only if

(a) $z <_L x$ for every $z \in X$.

(b) For every $y <_L x$ there is some $z \in X$ with $y \leq_L z \leq_X x$.

The notion of infimum is defined dually. A linear ordering $\langle L, <_L\rangle$ is Dedekind complete if and only if every bounded subset of L has a supremum in L. (A set $X \subseteq L$ is bounded, if and only if there is some $x \in X$ such that $z <_L x$ for all $z \in X$.)

Let $\langle L, <_L\rangle$ be a dense linear ordering without endpoints. Prove that there is a linear ordering $\langle K, <_K\rangle$ with the following properties:

(i) L is a dense subset of K.

(ii) $\langle K, <_K\rangle$ is Dedekind complete.

(iii) $\langle K, <_K\rangle$ is a dense linear ordering without endpoints.

Hint. A cut in $\langle L, <_L\rangle$ is a downward closed set without largest element. (For instance, $(-\infty, 0)$ and $(-\infty, \sqrt{2}) \cap \mathbb{Q}$ are cuts in $\langle \mathbb{Q}, <\rangle$, but in the latter case the cut cannot be identified as an interval in $\mathbb{Q}$.) Let K be the set of all cuts in $\langle L, <_L\rangle$ and $<_K$ be the sharp inclusion $\subset$. Prove that this gives the desired Dedeking completion.

3. Let $\langle K, <_K\rangle$ be a strict linear ordering and L be a dense subset of K; let $<_L$ be the induced linear ordering on L. Prove that $\langle L, <_L\rangle$ is a dense linear ordering if and only if $\langle K, <_K\rangle$ is a dense linear ordering.

4. Prove Cantor's theorem: If $\langle L, <_L \rangle$ and $\langle L', <_{L'} \rangle$ are two separable Dedekind complete dense linear orderings without endpoints then $\langle L, <_L \rangle$, $\langle L', <_{L'} \rangle$ are isomorphic.

Hint. Use the previous exercises. Start with picking an isomorphism between the corresponding countable dense subsets and try to extend it.

5. Let $\langle S, <_S \rangle$ be a Suslin line. On S define a binary relation $\sim$ by

$$x \sim y \iff \text{the interval } (x, y) \text{ is separable.}$$

Prove the following.

(a) $\sim$ is an equivalence relation on S and $<_S$ is $\sim$-invariant; this makes it possible to define the qotient $\langle S', <' \rangle = \langle S/\sim, <_S/\sim \rangle$.

(b) $\langle S', <' \rangle$ is a Suslin line.

(c) Every nonempy open interval in $\langle S', <' \rangle$ is nonseparable.

(d) $\langle S', <' \rangle$ is a dense linear ordering. Moreover, if we remove its endpoints (if they exist) the result will be a dense linear ordering without endpoints.

Remark. In general, if $\langle L, <_L \rangle$ is a linear ordering with endpoints, the removal of the endpoints does not guarantee that we obtain a linear ordering without endpoints. That is, the resulting linear ordering still may have endpoints. (Try to find a counterexample; more precisely find an example of a linear orering with endpoints such that if we remove the endpoints we obtain a linear ordering with endpoints, and such that this happens even if we iterate this procedure ω times, and after ω steps we end up with the empty set.) This makes (d) somehow special. Try to figure out why it happens in (d) and to formulate a reasonable general sufficient condition guaranteeing that if we remove the endpoints once then we obtain a linear ordering without endpoints.

6. Let $\langle T, <_T \rangle$ be a tree. Given two distinct maximal branches b, c in T, observe that there is a level T_α such that $b(\alpha) \neq c(\alpha)$. Here $b(\alpha)$ is the unique node in the intersection $b \cap T_\alpha$. For b, c as above let

$$\alpha(b, c) = \text{the least } \alpha \text{ such that } b(\alpha) \neq c(\alpha).$$

Let

$$B = \text{the set of all maximal branches in } T.$$

Assume that $<$ is some strict linear ordering on T and define a binary relation $\prec$ on B by

$$b \prec c \iff b(\alpha(b, c)) < c(\alpha(b, c)).$$

Prove that $\prec$ is a strict linear ordering on B.

7. Let $\langle \omega_1, <_T \rangle$ be a Suslin tree on ω_1 with splitting above every node (so $T = \omega_1$ here). Also let $<$ be the natural well-ordering of ordinals. Let $\langle B, \prec \rangle$ be constructed as in Exercise 6. Prove that $\langle B, \prec \rangle$ is a Suslin line.

Hint. To see the non-separability, notice that every maximal branch is countable. To see that $\langle B, \prec\rangle$ is c.c.c. notice that if $b, c \in B$ are such that $b \prec c$ then there is some $\xi \in \omega_1$ such that ξ is larger than all elements of $b - c$ and smaller than all elements of $c - b$ in the lexicographic ordering of the tree (see Week 2, Exercise 12).

8. Let $\langle T, <_T\rangle$ be a Suslin tree. Prove that $\langle T, <_T\rangle$ is not embeddable into real numbers, that is, there is no map $\sigma : T \to \mathbb{R}$ such that $x <_T y \implies \sigma(x) < \sigma(y)$ for all $x, y \in T$; here $<$ is the natural ordering of real numbers.

Hint. Assume for a contradiction there is such a map. To each $x \in T$ pick some $x' \in T$ such that x is the immediate predecessor of x'. Then pick some rational number $q_x \in (\sigma(x), \sigma(x'))$ and look what happens.

9. Let $\langle T, <_T\rangle$ be a Suslin tree with splittings above every node. Let

$$T' = \{\langle x, y\rangle \in T \times T \mid \mathrm{ht}_T(x) = \mathrm{ht}_T(y)\}$$

and $<_{T'}$ be the binary relation on T' defined from $<_T$ coordinate-wise, that is, for $\langle x, y\rangle$ and $\langle x', y'\rangle$ in T' we let

$$\langle x, y\rangle <_{T'} \langle x', y'\rangle \iff (x <_T y \,\&\, x' <_T y')$$

Prove that $\langle T' <_{T'}\rangle$ is an Aronszajn tree that is not Suslin and not embeddable into $\mathbb{R}$.

Hint. For simplicity, assume first that the splittings are immediate.

10. Let κ be regular. Prove that a Kurepa κ-tree exists if and only if there exists a Kurepa κ-family.

Hint. Given a Kurepa κ-tree $\langle \kappa, <_T\rangle$ (recall that we may let w.l.o.g. $T = \kappa$), consider the family of all cofinal branches through the tree. Conversely, given a Kurepa κ-family $\mathfrak{K} \subseteq \kappa$, consider $T = \{\chi_A \restriction \alpha \mid A \in \mathfrak{K} \,\&\, \alpha < \kappa\}$ where χ_A is the characteristic function of A.

11. Let κ be a singular cardinal of cofinality γ. Let $\langle \kappa_\xi \mid \xi < \gamma\rangle$ be a strictly increasing sequence of cardinals converging to κ. Prove that

$$\mathrm{card}(\prod_{\xi<\gamma} \kappa_\xi) = \kappa^\gamma$$

Hint. $\gamma \cdot \gamma = \gamma$.

12. Assume κ is a singular cardinal with uncountable cofinality and there are stationarily many $\mu < \kappa$ such that $2^\mu = \mu^{++}$. Imitate the proof of Silver's theorem to show that $2^\kappa \leq \kappa^{++}$.

13. Notice that the proof of Silver's theorem shows the following. Assume κ is a singular cardinal of uncountable cofinality γ and $\langle \kappa_\xi \mid \xi < \gamma\rangle$ is a normal sequence of cardinals converging to κ such that for every $\xi < \gamma$ we have $2^{\kappa_\xi} =$

κ_ξ^+. Assume further that $\mathcal{F}$ is an almost disjoint system of functions with domain γ such that for every $f \in \mathcal{F}$ there is a stationary set $S \subseteq \gamma$ such that for every $\xi \in S$ we have $f(\xi) < \kappa_\xi$. Then $\mathrm{card}(\mathcal{F}) \leq \kappa$.

14. Let κ be regular. Prove that there is an almost disjoint system of functions $f : \kappa \to \kappa$ of size κ^+.

Hint. Construct a sequence $\langle f_\xi \mid \xi < \kappa^+ \rangle$ by recrusion on ξ. Use a diagonal argument to move forward.

15. Prove that there is an almost disjoint family of functions $f : \omega \to \omega$ of size 2^ω. Prove also that the analogous conclusion holds for inaccessible cardinals κ in place of ω.

Hint. Look at characteristic functions of subsets of ω.

16. Amend Exercises 14 and 15 in that you construct families of almost disjoint sets. Given a cardinal κ we say that sets $X, Y \subseteq \kappa$ are almost disjoint iff $X \cap Y$ is bounded in κ.

(a) Show that if κ is regular then there is a family of κ^+ many almost disjoint subsets of κ.

(b) Show that there is a family of 2^ω many almost disjoint subsets of ω. Similarly, if κ is strongly inaccessible then there is a family of 2^κ many almost disjoint subsets of κ.

Hint. In (a) proceed by induction similarly as in Exercise 14. In (b) use the result of Exercise 15 and a suitable bijection between κ and the set of $\{0, 1\}$-functions with domain strictly shorter than κ. (Here you utilize the inaccessibility of κ.)