MATH 282B WINTER 2018 HOMEWORK 1

Target date: January 26, 2018

Rules: Write as efficiently as possible: Include all relevant points and think carefully what to write and what not. Use common sense to determine what is the appropriate amount of details for a course at this level. Quote any result from the lecture you are referring to; do not reprove the result. If the problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1 page) Assume the infinite Ramsey theorem $\omega \to (\omega)^n_k$ holds, and use it to prove the finite version of Ramsey theorm:

$$\text{If } m \in \omega \text{ then there is some } N \in \omega \text{ such that } N \to (m)^n_k. \tag{1}$$

For the proof, use a non-principal ultrafilter $\mathcal{U}$ over ω. Assume (1) fails. This gives you some $m \in \omega$ such that for every $N \in \omega$ you have a function $f : [N]^n \to k$ without a homogeneous set of size m. Use this and $\mathcal{U}$ to construct a function $f : [\omega]^n \to k$ without an infinite homogeneous set.

2. (1/2 page) Book, Problem 5.1.1 (There is a little overlap between this problem and Problem 3 below.)

3. (1/2 page) Let $\mathcal{L}$ be a language, $\mathcal{M}$ be an $\mathcal{L}$-structure, $(I, <)$ be a linear ordering, and $\mathcal{I} = (a_i \mid i \in I)$ be a system of indiscernibles for $\mathcal{M}$. Prove:

(i) If there are $i < j$ in I such that $a_i = a_j$ then $a_k = a_\ell$ for all $k, \ell \in I$.

(ii) No indiscernible is definable from other indiscernibles over $\mathcal{M}$, more precisely: If $i \in I$, $\varphi(u, v_1, \dots, v_\ell)$ is an $\mathcal{L}$-formula with all free variables displayed, and $j_1, \dots, j_\ell \in I$ are such that $j_h \neq i$ for $h = 1, \dots, \ell$ then φ does not define a_i from $a_{j_1}, \dots, a_{j_\ell}$ over $\mathcal{M}$. Formulate this precisely.

So the indiscernibility property is analogous to linear independence in vector spaces or algebraic independence in fields.

4. (1/2) page Let $\mathcal{L} = \{<\}$ be the language of linear orderings and T be the theory of linear orderings in the language $\mathcal{L}$. Prove that the class of all well-orderings is not elementary, that is, the property of being a well-ordering cannot be characterized by first-order statements. Precisely, prove that if Γ is a set of $\mathcal{L}$-formulae then it is not the case that

$$\mathcal{M} \text{is a well-ordering} \quad \text{iff} \quad \mathcal{M} \models \Gamma$$

whenever $\mathcal{M} \models T$.

5. (1/2 page) Let $\mathcal{L}$ be as in Exercise 4. Recall the following notions. If $(I, <)$ is a linear ordering, then a **cut** in $(I, <)$ is a pair of sets (I_1, I_2) such that

(i) $I = I_1 \cup I_2$

(ii) If $i \in I_1$ and $i' < i$ then $i' \in I_1$, i.e. I_1 is downward closed. Also if $i \in I_2$ and $i < i'$ then $i' \in I_2$, i.e. I_2 is upwnward closed.

(ii) $I_1 \cap I_2 = \varnothing$. (Equivalently, If $i \in I_1$ and $j \in I_2$ then $i < j$.)

A linear ordering $(I, <)$ is **Dedekind complete** iff every cut in $(I, <)$ is given by an element of I, that is, if I_1, I_2 is a cut then there is some $a \in I$ such that either $I_1 = (-\infty, a]$ and $I_2 = (a, \infty)$, or vice versa, i.e. $I_1 = (-\infty, a)$ and $I_2 = [a, \infty)$. Here the notions of interval, ∞ and $-\infty$ have the obvious meanings.

Use Skolem hulls to prove that the class of Dedeking complete linear orderings is not elementary. (This conclusion also follows from the fact that the theory of dense linear orderings without endpoints has quantifier elimination, but it is instructive to do a Skolem hull argument as well, as this one can be applied also in situations where quantifier elimination is not at hand.)