MATH 282B WINTER 2018 HOMEWORK 3

Target date: Friday, February 23

Rules: Write as efficiently as possible: Include all relevant points and think carefully what to write and what not. Use common sense to determine what is the appropriate amount of details for a course at this level. Quote any result from the lecture you are referring to; do not reprove the result. If the problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1/2 page) Let $\mathcal{M}$ be a saturated $\mathcal{L}$-structure of cardinality κ where $\mathcal{L}$ is a countable language and κ is an infinite cardinal. Assume $\varphi(v)$ is an $\mathcal{L}_M$-formula such that $\varphi(\mathcal{M})$ is infinite. Prove that $\mathsf{card}(\varphi(\mathcal{M})) = \kappa$.

2. (1/2 page) Let $\mathcal{L} = \{\dot{0}, \dot{1}, \dot{+}, \dot{\times}\}$ and let $\mathcal{N}$ be the standard model of arithmetic, that is, the domain of $\mathcal{N}$ is ω and the symbols $\dot{0}, \dot{1}, \dot{+}, \dot{\times}$ are interpreted in $\mathcal{N}$ as number 0, number 1, addition and multiplication, respectively.

Prove that $\mathsf{card}(S_1^{\mathcal{N}}(\varnothing)) = 2^{\aleph_0}$.

3. (2/3 page) (Book, Exercise 5.5.3) Prove that the theory of random graph has a Vaughtian Pair.

4. (1/2 page) Let $\mathcal{L}$ be the language of arithmetic and let $\mathcal{N}$ be the standard model of arithmetic, as in Exercise 2. Prove that if $\mathcal{N}'$ is an elementary extension of $\mathcal{N}$ then $(\mathcal{N}', \mathcal{N})$ is not a Vaughtian pair.

5. (1/2 page) Let $\mathcal{L}$ be a countable language, $\mathcal{M}$ be a countable $\mathcal{L}$-structure, and let $\mathcal{U}$ be a non-principal ultrafilter on ω. Prove that if $\mathcal{M}'$ is the ultrapower of $\mathcal{M}$ by $\mathcal{U}$ then $(\mathcal{M}', \mathcal{M})$ is not a Vaughtian pair. (Here we identify $\mathcal{M}$ with its image under the ultrapower embedding.)

6. (2/3 page) (Book, Exercise 5.5.4) Let T be a complete $\mathcal{L}$-theory, $\mathcal{M} \models T$ and $\vec{a}$ be a finite tuple of parameters from $\mathcal{M}$. Let $\vec{c}$ be a tuple of new constant symbols of the same length as $\vec{a}$. Consider the complete $\mathcal{L}_{\vec{c}}$-theory

$$T_{\vec{a}} = \{\varphi(\vec{c}) \mid \varphi(\vec{v}) \text{ is an } \mathcal{L}\text{-formula and } \mathcal{M} \models \varphi(\vec{a})\}$$

Notice that $T \subseteq T_{\vec{a}}$.

Prove that T has a Vaughtian pair iff $T_{\vec{a}}$ has.