Canonical Extension of Measures in Small Forcings

MATH 285A, April 2012.

In what follows we descibe how we can extend an elementary embedding $j : V \to M$ induced by an ultrafilter to an embeding π with domain $V[G]$, a forcing extension of V. We will only deal with the case of *small forcings*, in the sense that the forcing poset $\mathbb{P}$ belongs to V_κ, where κ is the critical point of the embedding j.

Definition 0.1. Let U be a nonprincipal ultrafilter on a set Z, with completeness κ, and let $j_U : V \to M$ be the induced elementary map. Let $\mathbb{P}$ be a forcing such that $\mathbb{P} \in V_\kappa$. Given G a $\mathbb{P}$-generic filter over V, we define $\pi_U : V[G] \to M[G]$ as follows:

$$\pi_U(\tau^G) := (j_U(\tau))^G.$$

It can be seen that the map above is a well-defined elementary map. Note also that $M[G]$ is a generic extension, since G is $\mathbb{P}$-generic over M.

The map π_U induces an ultrafilter $U' \subseteq \mathcal{P}(Z)$ in $V[G]$ defined as follows: letting $i^* = [id]_U \in M$ (where $id : Z \to Z$), we define:

$$U' := \{X \subseteq Z : i^* \in \pi_U(X)\}.$$

As we will see, U' is a κ-complete ultrafilter in $V[G]$.

Here are some properties of the map π_U:

Lemma 1. *Let $U, \mathbb{P}, \kappa$ be as in Definition 0.1. Then:*

(a) *For every $a \in V$, $(j_U(\check{a}))^G = j_U(a)$.*

(b) *$\pi_U \restriction_V = j_U$.*

(c) *$crit(\pi_U) = \kappa$.*

(d) *U' is κ-complete.*

(e) *π_U is equal to the elementary map $j_{U'} : V[G] \to N$ induced by U' (in particular $M[G] = Ult(V[G], U')$).*

Proof. (d) follows from (c), (c) follows from (b), and (b) follows from (a). To show (a): Since $\check{a} = \{1\} \times \{\check{b} : b \in a\}$, then by elementarity and absoluteness of the check construction (note also that $\mathbb{P} \in M$ and $crit(j_U) = \kappa$), we get that $j_U(a) = \{1\} \times \{\check{b} : b \in j_U(a)\} = \check{c}$, where $c = j_U(a) \in M \subseteq V$.

$$\begin{tikzcd}
V \arrow[r, "j_U"] \arrow[d, "id"] & M \arrow[d, "id"] \\
V[G] \arrow[r, "\pi_U = j_{U'}"] & M[G]
\end{tikzcd}$$

Now we prove (e). Let $j_{U'} : V[G] \to N$ be the embedding induced by U' (where $N = Ult(V[G], U')$). Let $k : N \to M[G]$ be the map defined by:

$$k([f]_{U'}) = \pi(f)(i^*).$$

It can be seen that k is a (well-defined) elementary embedding and that $j_{U'} \circ k = \pi$. We shall show that k is onto, and hence an isomorphism: since $M[G]$ and N are transitive, this will imply that $M[G] = N$ and k is the identity, and so $j_{U'} = \pi$.

$$\begin{tikzcd}
& M[G] \\
V[G] \arrow[ur, "\pi"] \arrow[r, "j_{U'}"] & N = \mathrm{Ult}(V[G], U') \arrow[u, "k"']
\end{tikzcd}$$

To show that $k : N \to M[G]$ is onto, let $b \in M[G]$, so that for some $g \in V$ with domain Z, $b = (\tau)^G$ (as evaluated in $M[G]$), where $\tau = [g]_U$ (for some $g \in V$ with domain Z) and

$$M \models \tau \text{ is a } \mathbb{P}\text{-name.}$$

So by absoluteness and since $M \subseteq V$, $V \models \tau$ is a $\mathbb{P}$-name, and by absoluteness of evaluations, τ^G as evaluated in $M[G]$ is equal to τ^G as evaluated in $V[G]$.

Therefore, since $j(\mathbb{P}) = \mathbb{P}$, by Los Theorem we get: there is $W \in U$ such that

$$\forall z \in W, \ \ V \models g(z) \text{ is a } \mathbb{P}\text{-name .}$$

So in $V[G]$ we can define a function F with domain Z such that for every $z \in Z$, $F(z) = g(z)^G$.

Claim: $k([F]_{U'}) = b$:
Proof of claim: for every $z \in W$ we have: $F(z) = g(z)^G$, so the set

$$A = \{z \in Z : F(z) = g(z)^G\}$$

belongs to U' (as U' extends U). So by definition of U', $i^* \in \pi(A)$, or in other words,

$$\pi(F)(i^*) = (\pi(g)(i^*))^G.$$

Since π extends j and $g \in V$, we get:

$$\pi(F)(i^*) = (j(g)(i^*))^G = ([g]_U)^G.$$

By definition, $k([F]_{U'}) = \pi(F)(i^*)$ and $b = ([g]_U)^G$, proving that $b \in Ran(k)$.

□