FORCING EXERCISES 1

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The exercises given here can be found, often along with solutions, in the literature. However, I strongly recommend that instead of looking into the literature you try to figure out the solutions on your own, using the given hints, or by asking for more hints. This way you learn much more.

None of the exercises/results in this series of exercises is due to me, unless explicitly stated. If you find out that some credits are missing or are inaccurate please notify me and I will make corrections.

The dependence in the exercises is vertical, that is, each exercise only uses results discussed in the preceding exercises. (At least I tried to list them in this order.

We follow the standard notation from books *Set Theory* by Jech, *Set Theory* by Kunen and the Handbook paper *Iterated forcing and elementary embeddings* by Cummings. Following Kunen, for a cardinal κ we let

$\mathsf{Fn}(I, J, \kappa) =$ the poset of all functions p of size $< \kappa$ with $\mathrm{dom}(p) \subseteq I$ and $\mathrm{rng}(p) \subseteq J$.

The ordering on $\mathsf{Fn}(I, J, \kappa)$ is reverse inclusion. Given cardinals κ, λ we also let

$$\mathsf{Add}(\kappa, \lambda) = \mathsf{Fn}(\lambda \times \kappa, 2, \kappa)$$

and

$$\mathsf{col}(\kappa, \lambda) = \mathsf{Fn}(\kappa, \lambda, \kappa).$$

Two forcing posets that are isomorphic give rise to the same generic extension. In fact, if $\pi : \mathbb{P} \to \mathbb{P}'$ is an isomorphism in the ground model and G, G' is a generic filter for $\mathbb{P}, \mathbb{P}'$ respectively then $\pi[G]$ is a generic filter for $\mathbb{P}'$ and $\pi^{-1}[G']$ is a generic filter for $\mathbb{P}$. (Check this!) If I' is equinumerous to I and J' is equinumerous to J then $\mathsf{Fn}(I', J', \kappa)$ is isomorphic to $\mathsf{Fn}(I, J, \kappa)$. For this reason we often take $\mathsf{Fn}(\lambda, 2, \kappa)$ as the poset $\mathsf{Add}(\kappa, \lambda)$.

A forcing poset $\mathbb{P}$ is

- κ-c.c. iff any antichain in $\mathbb{P}$ has size $< \kappa$.
- κ-closed iff any descending chain in $\mathbb{P}$ of size $< \kappa$ has a lower bound.
- (κ, λ)-distributive iff forcing with $\mathbb{P}$ does not any new function $f : \gamma \to \lambda$ whenever $\gamma < \kappa$. κ-distributivity means (κ, ∞)-distributivity.

Also, c.c.c. stands for ω_1-c.c. and σ-closed or countably closed stands for ω_1-closed. We will use the following two results

(a) If $\mathbb{P}$ is κ-c.c., A, B are sets in the ground model and $f : A \to B$ is a function in the generic extension then there is a function $F : A \to \mathcal{P}(B)$ such that
  - $f \subseteq F$. Equivalently, $f(a) \in F(a)$ for all $a \in A$.
  - $\mathsf{card}(F(a)) < \kappa$ for every $a \in A$.

(b) If $\mathbb{P}$ is κ-closed then $\mathbb{P}$ is κ-distributive, that is, $\mathbb{P}$ does not add any new function $f : \gamma \to \mathbf{On}$ whenever $\gamma < \kappa$ (hence $\mathbb{P}$ adds no new function $f : \gamma \to \mathbf{V}$ where $\gamma < \kappa$.)

The first couple of exercises are classical theorems. Try to work out the proofs on your own.

Exercise 1.1 Assume $\mathbb{P}$ is a forcing poset, A is a set in the ground model and $\dot{a}$ is a $\mathbb{P}$-name such that $p \Vdash \dot{a} \subseteq \check{A}$. Prove there is a $\mathbb{P}$-name $\dot{b}$ of the form

$$\bigcup_{a \in A} X_a \times \{\check{a}\}$$

where X_a is an antichain in $\mathbb{P}$ below p for each $a \in A$, and that $p \Vdash \dot{a} = \dot{b}$. □

You will need the above exercise to count the number of subsets of subsets of κ. The name $\dot{b}$ from the previous exercise is called a canonical name, or a nice name. Obviously, the notion of nice name can be generalized in a way that $\check{A}$ is replaced by some general name for a set in the generic extension, that is, not necessarily in the ground model.

A Δ-system is a family of sets $\mathcal{A}$ such that there is a set r satisfying $x \cap y = r$ for all $x, y \in \mathcal{A}$ with $x \neq y$. A frequently used tool for computing chain conditions is the Δ-system lemma: Let $\mu < \theta$ be cardinals such that θ is regular and $\alpha^{<\mu} < \theta$ for all $\alpha < \theta$. If $\mathcal{A}$ is a family of sets such that $\mathsf{card}(\mathcal{A}) = \theta$ and each element of $\mathcal{A}$ has size $< \mu$ then there is a Δ-system $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathsf{card}(\mathcal{B}) = \theta$.

Exercise 1.2. Let κ be a regular cardinal. Use the Δ-system lemma to establish the following.

(a) $\mathsf{Add}(\kappa, \lambda)$ is $(\kappa^{<\kappa})^+$-c.c.
(b) If $\lambda = 2^\kappa$ then $\mathsf{col}(\kappa, \lambda)$ is λ^+-c.c. Similarly, $\mathsf{col}(\kappa^+, \lambda)$ is λ^+-c.c.
(c) Both $\mathsf{Add}(\kappa, \lambda)$ and $\mathsf{col}(\kappa, \lambda)$ are κ-closed.

Hint. Use the Δ-system lemma. □

Exercise 1.3 Let κ be regular cardinal and λ be a cardinal such that $\mathsf{cf}(\lambda) > \kappa$. Assume GCH.

(a) Let $\mathbb{P} = \mathsf{Add}(\kappa, \lambda)$. Show that forcing with $\mathbb{P}$ preserves all cardinals an cofinalities, and in the genric extension we have $2^\kappa = \lambda$.
(b) Let $\lambda = 2^\kappa$ and $\mathbb{P} = \mathsf{col}(\kappa^+, \lambda)$. Prove that forcing with $\mathbb{P}$ preserves all cardinals and cofinalities except those in the interval $(\kappa^+, \lambda]$, and in the generic extension we have $2^\kappa = \kappa^+$.
(c) Let $\lambda = 2^\kappa$ and $\mathbb{P} = \mathsf{col}(\kappa, \lambda)$. Prove that forcing with $\mathbb{P}$ preserves all cardinals and cofinalities except those in the interval$(\kappa, \lambda]$, and in the generic extension we have $2^\kappa = \kappa^+ \lambda^+$.
(d) In the situation from (c): Prove that if α is an ordinal such that in the ground model we have $\kappa \leq \mathsf{cf}(\alpha) \leq \lambda$ then in the generic extension we have $\mathsf{cf}(\alpha) = \kappa$.

Hint. (d) First prove this for regular α. □

Exercise 1.4 Show that forcing with $\mathsf{Add}(\aleph_\omega, 1)$ collapses all cardinals $\aleph_\alpha$ for $\alpha \leq \omega$ to ω.

Hint. For each $n \in \omega$ find a way of coding a surjection of ω onto ω_n into the generic subset of $\aleph_\omega$. For this use the density argument. □

Exercise 1.5 Prove the following.

(a) Forcing with $\mathsf{col}(\omega, \aleph_\omega)$ turns $\aleph_{\omega+1}$ of the ground model into ω_1 of the generic extension.

(b) Assume CH, or more generally, $2^\omega < \aleph_\omega$. Forcing with $\mathsf{col}(\omega_1, \aleph_\omega)$ keeps ω_1 of the ground model a cardinal in the generic extension. However, all cardinals $\aleph_\alpha$ will be collapsed to ω_1 for $\alpha \in (2, \omega+1]$. (This is of course trivial for $\alpha < \omega + 1$.)

Hint. Regarding (b): Look at $(\aleph_\omega)^\omega$ of the ground model; here apply König's lemma. □

Recall that a $\diamondsuit_\kappa(E)$-sequence is a sequence $\langle a_\alpha \mid \alpha \in E\rangle$ such that $a_\alpha \subseteq \alpha$ for all $\alpha \in E$, and for every $X \subseteq \kappa$ there are stationarily many $\alpha \in E$ with $X \cap \alpha = a_\alpha$. We say that $\diamondsuit_\kappa(E)$ holds if such a sequence exists. We write $\diamondsuit_\kappa$ for $\diamondsuit_\kappa(\kappa)$ and $\diamondsuit$ for $\diamondsuit_{\omega_1}$. Obviously $E \subseteq E'$ implies $\diamondsuit_\kappa(E) \Rightarrow \diamondsuit_\kappa(E')$. If $\diamondsuit_\kappa$ holds then κ is necessarily regular and $2^{<\kappa} = \kappa$. (Check all of this!)

Similarly we define a $\diamondsuit'_\kappa(E)$-sequence to be a sequence $\langle A_\alpha \mid \alpha \in E\rangle$ such that $A_\alpha \subseteq \mathcal{P}(\alpha)$ and $\mathsf{card}(A_\alpha) \leq \kappa$ for all $\alpha \in E$, and for every $X \subseteq \kappa$ there are stationarily many $\alpha \in E$ with $X \cap \alpha \in A_\alpha$. A theorem of Kunen says that $\diamondsuit'_\kappa(E)$ is equivalent to $\diamondsuit_\kappa(E)$; one of these implications is trivial.

Exercise 1.6 Let $\mathbb{P}$ be the forcing defined as follows.

- Conditions are functions p such that $\mathrm{dom}(p)$ is a countable ordinal, and $p(\alpha) \subseteq \alpha$ for all $\alpha \in \mathrm{dom}(p)$.
- Ordering is end-extension.

Prove that $\mathbb{P}$ is countably closed and that $\diamondsuit$ holds in the generic extension by $\mathbb{P}$.

Hint. This is straightforward, but takes some work. If G is the generic filter then $\bigcup G$ is the $\diamondsuit$-sequence. Pick a name $\dot{X}$ for a subset of ω_1 and a name $\dot{C}$ for a club in ω_1, and using the forcing relation find a condition p and an ordinal α such that $\alpha \in \mathrm{dom}(p)$ and $p \Vdash (\check{\alpha} \in \dot{C} \;\&\; \dot{X} \cap \check{\alpha} = \widecheck{p(\alpha)})$. □

Exercise 1.7 Assume $\diamondsuit$ holds.

(a) If $\mathbb{P}$ is a countable poset then $\diamondsuit$ holds in the generic extension by $\mathbb{P}$ if and only if it holds in the ground model.

(b) Assume $\diamondsuit$ holds in the ground model. There is a c.c.c. poset such that $\diamondsuit$ fails in the generic extension by $\mathbb{P}$.

(c) If $\diamondsuit$ holds in the ground model and $\mathbb{P}$ is a countably closed poset then $\diamondsuit$ holds in the generic extension by $\mathbb{P}$.

(d) Show that it is consistent to have $\diamondsuit + 2^{\omega_1} = \omega_7$.

Hint. (a) Use Kunen's theorem. Given a $\diamondsuit$-sequence in the ground model, show that this sequence gives rise to a $\diamondsuit'$-sequence in the generic extension. On the other hand, given a $\diamondsuit$-sequence in the generic extension use the countability of $\mathbb{P}$ to construct a $\diamondsuit'$-sequence in the ground model. Both take some work.

(b): Look at the posets considered so far.

(c): Let $a = \langle a_\alpha \mid \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence in the ground model. Show that this sequence is a $\diamondsuit$ sequence in the generic extension. Assuming this fails, pick a name $\dot{X}$ for a subset of ω_1, a name $\dot{C}$ for a club subset of ω_1, and a condition p such that $p \Vdash (\forall \xi \in \dot{C})(\dot{X} \cap \xi \neq \check{a}_\xi)$. Try to get a contradiction by constructing a set $X \subseteq \omega_1$ and a closed unbounded $C \subseteq \omega_1$ both in the ground model such that $X \cap \alpha \neq a_\alpha$ for all $\alpha \in C$. □

Given a cardinal κ and a set A the poset $\mathsf{Coll}(\kappa, A)$ is defined as follows.

- Conditions are functions $p : a \to \bigcup A$ such that $a \in [\kappa \times A]^{<\kappa}$ and
$$p(\xi, a) \in a \text{ whenever } (\xi, a) \in \mathrm{dom}(p).$$
- Ordering is by reverse inclusion.

$\mathsf{Coll}(\kappa, A)$ is called the Lévy collapse of A to κ. It adds a surjection of κ onto a for every element $a \in A$. If κ is regular then $\mathsf{Coll}(\kappa, A)$ is κ-closed, so it preserves all cardinals $\leq \kappa$. (Check all of this!) An important special case is where $A = \lambda$ for a cardinal $\lambda \geq \kappa$. It is a convention to write $\mathsf{Coll}(\kappa, < \lambda)$ instead of $\mathsf{Coll}(\kappa, \lambda)$. Write down the conditions of $\mathsf{Coll}(\kappa, < \lambda)$.

Exercise 1.8 Let λ be a regular uncountable cardinal.

(a) Show that $\mathsf{Coll}(\omega, < \lambda)$ is λ-c.c. and in the generic extension we have $\lambda = \omega_1$.
(b) Show that if λ is strongly inaccessible then $\mathsf{Coll}(\omega_1, < \lambda)$ is λ-c.c. and in the generic extension we have $\lambda = \omega_1$.
(c) Show that in the generic extension by $\mathsf{Coll}(\omega, < \aleph_\omega)$ the ground model cardinal $\aleph_{\omega+1}$ becomes ω_1.
(d) Assume CH or more generally that $2^\omega < \aleph_\omega$. Show that in the generic extension by $\mathsf{Coll}(\omega_1, < \aleph_\omega)$ the ground model cardinal $\aleph_{\omega+1}$ is collapsed to ω_1.

Hint. For (a),(b) use the Δ-system lemma. (c) is an instance of (a) and is only given to see the contrast with (d); here (d) is similar to (b) in Exercise 1.5. □

A $\Diamond^*_\kappa$-sequence is a sequence of the form $\langle A_\alpha \mid \alpha < \kappa \rangle$ such that $A_\alpha \subseteq \mathcal{P}(\alpha)$ and $\mathsf{card}(A_\alpha) \leq \kappa$ for every $\alpha < \kappa$, and for all $X \subseteq \kappa$ there is a club subset $C \subseteq \kappa$ such that $X \cap \alpha \in A_\alpha$ for all $\alpha \in C$. $\Diamond^*_\kappa$ is the statement that such a sequence exists. By the theorem of Kunen, $\Diamond^*_\kappa$ implies $\Diamond_\kappa(E)$ for all stationary $E \subseteq \kappa$.

Exercise 1.9 Assume $\kappa < \lambda$ are such that κ is regular and λ is inaccessible. Show that in the forcing extension with $\mathsf{Coll}(\kappa, < \lambda)$ we have $\lambda = \kappa^+$ and $\Diamond^*_{\kappa^+}$.

Hint. Use the chain condition to capture initial segments of canonical names of subsets of λ. □

Exercise 1.10 Let κ be strongly inaccessible.

(a) Prove that if we force with $\mathsf{Coll}(\omega, < \kappa)$ then CH holds in the generic extension, although we add κ real numbers. (This is weaker than the result in Exercise 1.9, and the argument is easier.)
(b) Prove that if we force with $\mathsf{Coll}(\omega_1, < \kappa)$ then CH holds in the generic extension. Compare this to (a). Also pay attention to what ω_1 of the generic extension is in either case.
(c) Force with $\mathsf{Coll}(\omega_2, < \kappa)$. Can we conclude that CH holds in the generic extension?
(d) Assume $\lambda = 2^\omega$ is regular. Determine if CH hold in the generic ext extension if we force with the following posets: $\mathsf{col}(\omega, \lambda)$, $\mathsf{Coll}(\omega, < \lambda)$, $\mathsf{col}(\omega_1, \lambda)$, $\mathsf{Coll}(\omega_1, < \lambda)$. How do the conclusions look if we assume that λ is singular?