FORCING EXERCISES 2

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The exercises given here can be found, often along with solutions, in the literature. However, I strongly recommend that instead of looking into the literature you try to figure out the solutions on your own, using the given hints, or by asking for more hints. This way you learn much more.

None of the exercises/results in this series of exercises is due to me, unless explicitly stated. If you find out that some credits are missing or are inaccurate please notify me and I will make corrections.

The dependence in the exercises is vertical, that is, each exercise only uses results discussed in the preceding exercises. (At least I tried to list them in this order.

Recall that a tree is a partially ordered set $(T, <_T)$ such that for each $t \in T$ the set of all $<_T$-predecessors of t is well-ordered under $<_T$. The order type of this set under $<_T$ is the height of t in $(T, <_T)$; we denote it by $\mathsf{ht}_T(t)$. We write briefly T instead of $(T, <_T)$ when $<_T$ is clear from the context. The α-th level of T, denoted by T_α, is the set of all $t \in T$ with $\mathsf{ht}_T(t) = \alpha$. We also let $T_{<\alpha}$ be the union of all T_ξ where $\xi < \alpha$. The least α such that $T_\alpha = \varnothing$ is called the height of T and denoted by $\mathsf{ht}(T)$. Given a cardinal κ, a tree of height κ is is a κ-tree if all levels of T have size $< \kappa$. A tree of height κ is pruned iff there are κ nodes above any node in T. Given any κ-tree T one can a subtree of T that is pruned. A branch through T is any $b \subseteq T$ that is downward closed under $<_T$ and on which the tree ordering $<_T$ is linear. A branch b through T is maximal if there is no branch b' properly extending b. A branch b through T is cofinal if it is ordered under $<_T$ in order type $\mathsf{ht}(T)$. If T is a κ-tree then $\mathsf{card}(T) = \kappa$, so we may without loss of generality assume that $(T, <_T) = (\kappa, <_T)$, that is, the set of nodes is κ.

Exercise 2.1. (Silver) Assume T is an ω_1-tree and $\mathbb{P}$ is a countably closed forcing. Prove that $\mathbb{P}$ does not add any new cofinal branch through T, that is: If b is a cofinal branch through T in the generic extension then b is actually in the ground model.

Hint. If $p \in \mathbb{P}$ is a condition and $\dot{b}$ is a $\mathbb{P}$-name such that

$$p \Vdash \dot{b} \text{ is a cofinal branch through } T \ \&\ \dot{b} \text{ is not in the ground model}$$

(make the second statement in the formula above formally correct!) then for cofinally many $\alpha < \omega_1$ there are $q_0, q_1 \leq p$ and $\xi_0 \neq \xi_1$ such that $q_0 \Vdash \xi_0 \in \dot{b} \cap \check{T}_\alpha$ and $q_1 \Vdash \xi_1 \in \dot{b} \cap \check{T}_\alpha$. Try to develop this idea and get a contradiction by showing that T has an uncountable level. □

Recall that given a regular cardinal κ, an Aronszajn κ-tree is a κ-tree with no cofinal branch. A κ-Suslin tree is an Aronszajn κ-tree such that each antichain in the tree has size $< \kappa$. If T is a κ-tree such that there is a split above every node (not necessarily an immediate one) then the requirement that all antichains in T are of size $< \kappa$ guarantees that T is κ-Suslin (check this!).

Exercise 2.2. Assume θ is regular and $\kappa < \theta$ is a cardinal. Let T be a tree of height θ all of whose levels are of size $< \kappa$. Prove that T has a cofinal branch. (So there is no forcing here.)

Hint. Without loss of generality T is pruned and there is a splitting above every node. (Check this!) If κ is regular, find some level T_α such that $\mathsf{cf}(\alpha) = \kappa$, and for every $\bar{\alpha} < \alpha$ and $t \in T_{\bar{\alpha}}$ there is a splitting above t at some level below T_α. Try to modify/apply this argument also in the case where κ is singular. □

Exercise 2.3. (Magidor-Shelah) Assume κ is a cardinal of cofinality ω and T is a κ^+-tree.

(a) Assume $\mathbb{P}$ is a forcing, $\dot{b}$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$ is a condition that forces the statement "$\dot{b}$ a new cofinal branch through $\check{T}$". Prove that there is a level T_α, conditions $q_i \leq p$, and mutually distinct nodes ξ_i for $i < \kappa$ such that $q_i \Vdash \xi_i \in \dot{b} \cap (T_\alpha)\check{}$. Notice that the same is then true of all $\alpha' > \alpha$.

(b) Prove that if $\mathbb{P}$ is countably closed then $\mathbb{P}$ does not add any new cofinal branch through T.

Hint. (a) Assuming the contrary, try to construct a subtree T' of T of height κ^+ all of whose levels are of size $< \kappa$. Do this by looking at all nodes in T that may be on the branch $\dot{b}^G$ for some generic filter G. Then apply the ideas from the proofs of Exercises 2.1 and 2.2 to get a contradiction.

(b) Combine the result from (a), the idea in Exercise 2.1 and König's lemma from cardinal arithmetic. □

A forcing notion $\mathbb{P}$ has the κ-Knaster property if and only if for every sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$ of conditions in $\mathbb{P}$ there is an $A \subseteq \kappa$ of size κ such that $\{p_\alpha \mid \alpha \in A\}$ is a set of pairwise compatible conditions. Notice that if $\mathbb{P}$ has the κ-Knaster property then $\mathbb{P}$ is κ-c.c. Notice also that a Suslin tree is a c.c.c. forcing which does not have the ω_1-Knaster property. (Assuming it has an ω_1-Knaster property, try to construct a cofinal branch.)

The following exercise goes along the lines of the previous ones, but there are two major differences: We relax the assumption on T in that we allow its levels to be arbitrarily large; on the other hand we make a restrictive assumption that T has no cofinal branches in the ground model. In the previous exercises we did not make such an assumption, that is, T was allowed to have cofinal branches.

Exercise 2.4. (Kunen-Tall) Assume κ is a cardinal and T is a tree of height κ without cofinal branches. Let $\mathbb{P}$ is a poset with κ-Knaster property. Prove that $\mathbb{P}$ does not add any cofinal branch through T.

Hint. Assuming p forces that $\dot{b}$ is a new cofinal branch through T for each α pick a condition $p_\alpha \leq p$ which determines the node in $\dot{b} \cap T_\alpha$. Now try to construct a cofinal branch in the ground model. □

Using this exercise, try to find another argument why a Suslin tree is a c.c.c. poset which does not have the ω_1-Knaster property.

Exercise 2.5. Let κ be a regular cardinal and I, J be sets. Prove that $\mathsf{Fn}(I, J, \kappa)$ has $2^{<\kappa} \cdot \mathsf{card}(J)^+$-Knaster property.

Hint. Use the Δ-system argument. □

Exercise 2.6. (Jech) Let $\mathbb{P}$ be the posed defined as follows.

- Conditions are countable trees $p = (\alpha_p, \leq_p)$ where α_p is a countable ordinal satisfying:
  - If ξ is node in p then either ξ has two immediate successors in p, or else p has successor height and $\beta + 1$ and ξ is in the last level p_β.
  - If $\beta < \mathsf{ht}(p)$ is a limit ordinal and b is a branch in p of height β then b has at most one successor on the level p_β.
  - For every $\xi \in p$ and every β such that $\mathsf{ht}_p(x) \leq \beta < \mathsf{ht}(p)$ there is a node ζ in p such that $\xi <_p \zeta$ and $\mathsf{ht}_p(\zeta) = \beta$.
- Ordering is by end-extension. Precisely, $p \leq q$ iff the following hold.
  - $\alpha_p \supseteq \alpha_q$.
  - $<_q = <_p \cap (\alpha_q \times \alpha_q)$.
  - If $\xi \in \alpha_p$, $\zeta \in \alpha_p - \alpha_q$, and ξ, ζ are comparable with respect to $<_p$ then $\xi <_p \zeta$.

Let G be generic for $\mathbb{P}$,

$$\begin{aligned} \alpha &= \bigcup\{\alpha_p \mid p \in G\} \\ < &= \bigcup\{<_p \mid p \in G \end{aligned}$$

Then $\alpha = \omega_1$ and $T = (\omega_1, <)$ is an ω_1-Suslin tree on ω_1 in the generic extension. The proof consists of several steps.

(a) Prove that $\mathbb{P}$ is ω_1-closed.
(b) Show that $\alpha = \omega_1$ and T is an ω_1-tree on ω_1.
(c) Prove that every antichain in T is at most countable. This suffices for the conclusion that T is a Suslin tree. (Why?)

Hint. For (b), this is a density argument. The argument uses induction on α_p. Regarding (c). Assume $\dot{T}$ is a name for the tree defined above and

$$p \Vdash \dot{a} \text{ is a maximal antichain in } \dot{T}.$$

Using the fact that $\mathbb{P}$ is ω_1-closed construct in countably many steps a condition $q \leq p$ and a set c in the ground model such that $q \Vdash \alpha_q \cap \dot{a} = \check{c}$, and also $q \Vdash \check{c}$ is a maximal antichain in $\check{q}$. Then extend q to a condition $q' \leq q$ by adding the α_q-th level to q such that every node on this α_q-th level is above some element of c. (This procedure is called "sealing off" antichains; notice the resemblance with the construction of a Suslin tree from a $\diamondsuit$-sequence.) $\square$

Exercise 2.7. (Tennenbaum) Let $\mathbb{P}$ be a poset defined as follows.

- Conditions are finite trees $p = (a_p, <_p)$ satisfying:
  - a_p is a finite subset of ω_1.
  - $<_p$ is a tree ordering on a_p consistent with the natural ordering on ordinals, that is, for $\alpha, \beta \in a_p$ we have
  $$\alpha <_p \beta \implies \alpha < \beta.$$
- Ordering is defined by letting $p \leq q$ iff q is an induced subtree of p, that is:
  - $a_p \supseteq a_q$.
  - $<_q = <_p \cap (a_q \times a_q)$.

Prove the following facts.

(a) $\mathbb{P}$ is c.c.c.

(b) Let G be a generic filter for $\mathbb{P}$,

$$\begin{aligned} a &= \bigcup\{a_p \mid p \in G\}, \\ <_T &= \bigcup\{<_p \mid p \in G\}, \end{aligned}$$

and $T = (a, <_T)$. Prove that $a = \omega_1$ and T is a tree on ω_1 of height $\leq \omega_1$.

(c) Prove that there is a splitting above every node in T; although this splitting is not necessarily immediate.

(d) Prove that every antichain in T is countable. Conclude that T is a Suslin tree.

Hint. For (a), use the Δ-system argument. Regarding (d), assuming there is an antichain in T of size ω_1 let $\dot{T}$ be a name for the tree T constructed as above, $p \in \mathbb{P}$, and $\dot{A}$ be a $\mathbb{P}$-name such that

$$p \Vdash \dot{A} \text{ is an uncountable antichain in } \dot{T}.$$

Find sequences $\langle p_\xi \mid \xi < \omega_1 \rangle$ and $\langle \alpha_\xi \mid \xi < \omega_1 \rangle$ such that $p_\xi \leq p$, $\alpha_\xi \in a_{p_\xi}$, and $p_\xi \Vdash \check{\alpha}_\xi \in \dot{A}$. Then apply the Δ-system argument. □

Exercise 2.8. Starting from arbitrary ground model, figure out how to force to get a generic extension satisfying the following.

(a) $\mathsf{CH} +$ "There is a Suslin tree".

(b) $2^\omega \geq \aleph_\alpha +$ "There is a Suslin tree" for an ordinal α given in advance such that $\mathsf{cf}(\alpha) \neq \omega$. ($\alpha$ may be a successor ordinal.)

Hint. Use some of the previous exercises. □

Recall that given a cardinal μ and letting $\kappa = \mu^+$, a κ-Kurepa tree is a κ-tree with at least κ^+ many cofinal branches. So levels of the tree have size at most μ. If κ is strongly inaccessible then this definition of a κ-Kurepa tree is not suitable, as the complete binary tree of height κ would be a κ-Kurepa tree according to this definition. Hence for regular limit cardinals (= weak inaccessibles) we define a κ-Kurepa tree to be a κ tree with at least κ^+ many cofinal branches such that for each $\alpha < \kappa$ the level T_α is of cardinality $\leq \mathsf{card}(\alpha)$.

Exercise 2.9. Let κ be a strongly inaccessible cardinal and T be a complete binary tree of height κ. Show that if we force with $\mathsf{Coll}(\omega, <\kappa)$ then T is an ω_1-Kurepa tree in the generic extension.

However, it is not necessary to use an inaccessible cardinal to build a model with a Kurepa tree. (Recall that there is a Kurepa tree in Gödel's constructible universe $\mathbf{L}$.)

Exercise 2.10. Let $\mathbb{P}$ be a forcing defined as follows.

- Conditions are pairs (p, f) where:
  - p is as in Exercise 2.6.
  - f is an injective function such that $\mathrm{dom}(f)$ is a countable subset of ω_2, and for every $\alpha \in \mathrm{dom}(f)$ the value $f(\alpha)$ is a cofinal branch through the tree p.
- Ordering is defined by $(p, f) \leq (q, g)$ iff
  - $p \leq q$ where $\leq$ is as in Exercise 2.6, and

– $\mathrm{dom}(f) \supseteq \mathrm{dom}(g)$ and for every $\alpha \in \mathrm{dom}(g)$ the branch $g(\alpha)$ is an initial segment of $f(\alpha)$.

Prove the following facts.

(a) $\mathbb{P}$ is ω_1-closed.
(b) If CH holds then $\mathbb{P}$ is ω_2-c.c.
(c) If CH holds then in any generic extension with $\mathbb{P}$ there is an ω_1-Kurepa tree.

The above statements are proved using standard arguments of the kind used in the previous exercises. □

Exercise 2.11. Let κ be a strongly inaccessible cardinal. Following Exercise 2.10, define a forcing poset for adding a κ-Kurepa tree, formulate the analog statements to (a)–(c) in Exercise 2.10, and prove them. □

Given a poset $\mathbb{P}$, a set $A \subseteq \mathbb{P}$ is directed if and only if for any $a, a' \in A$ there is some $b \in A$ such that $b \leq a, a'$. A forcing poset $\mathbb{P}$ is κ-directed closed if and only if every directed set $A \subseteq \mathbb{P}$ of cardinality $< \kappa$ has a lower bound in $\mathbb{P}$, that is, there is an element $b \in \mathbb{P}$ such that $b \leq a$ for all $a \in A$. If $\mathbb{P}$ is κ-directed closed then $\mathbb{P}$ is κ closed. (Prove this.)

Exercise 2.12. Prove the following.

(a) If κ is a regular cardinal and I, J are sets then $\mathsf{Fn}(I, J, \kappa)$ is κ-directed closed. In particular, the posets $\mathsf{Add}(\kappa, \lambda)$ and $\mathsf{col}(\kappa, \lambda)$ are κ-directed closed.
(b) Given a regular cardinal κ and a set A the poset $\mathsf{Coll}(\kappa, A)$ is κ-directed closed.
(c) If $\mathbb{P}$ is ω_1-closed then $\mathbb{P}$ is ω_1-directed closed.

The proofs of the above clauses are straightforward. □

For regular κ larger than ω the property of being κ-closed does not necessarily imply the property of being κ-directed closed.

Exercise 2.13. Prove that the poset from Exercise 2.11 is κ-closed but not κ-directed closed.

Hint. Assuming the poset is κ-directed closed try to get a pair (p, f) where the tree p has a level of too large cardinality, but which would be nevertheless obliged to be a condition in $\mathbb{P}$. □

Given a regular cardinal, a poset $\mathbb{P}$ has the κ-approximation property at a cardinal λ if and only if for every $\mathbb{P}$-name $\dot{x}$ such that $\Vdash_{\mathbb{P}} \dot{x} \subseteq \lambda$ the following holds: If $\Vdash_{\mathbb{P}}$ "$\dot{x} \cap \check{a}$ is in the ground model" whenever $a \in [\lambda]^{<\kappa}$ is in the ground model then $\Vdash_{\mathbb{P}}$ "$\dot{x}$ is in the ground model". A poset $\mathbb{P}$ has the κ-approximation property if and only if $\mathbb{P}$ has the κ-approximation property at all cardinals λ.

Notice that for instance $\mathsf{Add}(\omega, \lambda)$ has the ω_1-approximation property, but this is a trivial fact, as the hypothesis in the definition of the ω_1-approximation property fails in the case of $\mathsf{Add}(\omega < \lambda)$. All known forcings with the approximation property that are interesting are quite complicated and are beyond of the level of difficulty of this sheet of exercises. Nevertheless, there are some interesting exercises at this level of difficulty.

Exercise 2.14. Assume $\mathbb{P}$ is a tree of height κ where κ is regular, and $\mathbb{P}$ is a poset with the κ-approximation property. Prove that $\mathbb{P}$ does not add any new cofinal branch through T.

Hint. Apply the relevant definitions directly. ◻

Exercise 2.15. Assume $\mathbb{P}$ is a poset, $\kappa \leq \lambda$ are cardinals, κ is regular, $p \in \mathbb{P}$, and $\dot{x}$ is a $\mathbb{P}$-name such that:

- $p \Vdash \dot{x} \subset \check{\lambda}$.
- $p \Vdash \dot{x}$ is not in the ground model.
- $p \Vdash \dot{x} \cap \check{a}$ is in the ground model whenever $a \in [\lambda]^{<\kappa}$ is in the ground model.

Prove that for every $a \in [\lambda]^{<\kappa}$ in the ground model there are a set $b \in [\lambda]^{<\kappa}$ in the ground model, conditions $p', p'' \leq p$, and sets $y \neq z$ in the ground model such that $b \supseteq a$, $p' \Vdash \dot{x} \cap \check{b} = \check{y}$, and $p'' \Vdash \dot{x} \cap \check{b} = \check{z}$.

Hint. Modify the argument from Hint to Exercise 2.1 suitably. ◻

The following exercise is a warm-up for Exercise 2.17.

Exercise 2.16. Let T be a tree of height ω_1 such that there is a splitting above every node in T. Show that $T \times T$ has a subset of size ω_1 of pairwise incompatible elements. If we turn the trees upside down and view them as forcing posets, then this is equivalent to saying that $T \times T$ does not have the c.c.c.

Hint. Look at nodes $t_\alpha, t'_\alpha, t''_\alpha$ such that t'_α, t''_α are above t_α in T and the height of t_α in T is larger than the heights of all $t'_{\bar{\alpha}}, t''_{\bar{\alpha}}$ for all $\bar{\alpha} < \alpha$. ◻

Exercise 2.17. (Unger, inspired by an argument of Mitchell) Assume κ is regular and $\mathbb{P}$ is a poset such that such that $\mathbb{P} \times \mathbb{P}$ has the κ-c.c. Prove that $\mathbb{P}$ has the κ-approximation property.

Hint. Aiming for a contradiction, assume $\mathbb{P}$ does not have the κ-approximation property at λ, and this is witnessed by a name $\dot{x}$. Construct inductively in κ steps conditions $p_\alpha, p'_\alpha, p''_\alpha$ in $\mathbb{P}$ and sets $a_\alpha \in [\lambda]^{<\kappa}$ such that $a_\alpha < a_\beta$ whenever $\alpha < \beta$, $p'_\alpha, p''_\alpha \leq p_\alpha$, the condition p_α decides the value of $\dot{x} \cap \check{a}_\alpha$, and the conditions p'_α, p'_α decide the value of $\dot{x} \cap \check{a}_{\alpha+1}$ in two inconsistent ways. Follow the analogy with Exercise 2.16. ◻

Recall that if κ is a cardinal and T is a κ^+-tree then T is special if and only if there is a function $f : T \to \kappa$ such that $f(t) \neq f(t')$ whenever t, t' are comparable nodes in T. Equivalently, $f \restriction b$ is injective whenever b is a branch in T. Notice that a special tree of height κ^+ does not have a cofinal branch; in particular, a special κ-tree (that is, a tree whose levels are of size $\leq \kappa$) is always Aronszajn. Notice also that if T is a special tree of height κ^+ then in any generic extension (in fact in any extension) where κ^+ of the ground model is not collapsed T is a special tree with specializing function f. Hence there are no cofinal branches through T even in the extension. The following exercise shows that any tree of height ω_1 without cofinal branches can be turned into a special tree via a very nice forcing (of course there cannot be cofinal branches if we are hoping to specialize the it).

Exercise 2.18. Let T be a tree of height ω_1 without cofinal branches. Let $\mathbb{P}$ be a poset defined as follows.

- Conditions are finite functions p satisfying the following.
  - $\mathrm{dom}(p)$ is a finite set of nodes of T and $\mathrm{rng}(p) \subseteq \omega$.
  - If $t <_T t'$ are in $\mathrm{dom}(p)$ then $p(t) \neq p(t')$.

- Ordering is by end-extension.

Prove the following.

(a) $\mathbb{P}$ is c.c.c.

(b) $\mathbb{P}$ adds a specializing function for T.

Hint. Regarding (a). Assuming $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ is an antichain in $\mathbb{P}$, thin out the sequence so that the domains of p_α have the same finite number of elements, say n, and constitute a Δ-system with root r; let $s_\alpha = \mathrm{dom}(p_\alpha) - r$. Argue that for each α there are at most countably many α' such that some element of $s_{\alpha'}$ is below some element of s_α in the tree ordering $<_T$. Denote S be the set of all these s_α, and let U be a uniform ultrafilter on $\mathcal{P}(S)$. Also fix an enumeration of each $s \in S$, and denote the i-th element of s under this enumeration by $t(s, i)$. Using the above conclusions, show that to each $s \in S$ we can assign a pair (t_s, i_s) where $t_s \in s$ and $i_s \in \{1, \ldots, n\}$ are such that the set $X_s = \{s' \in S \mid t_s <_T t(s', i_s)\} \in U$. Show that there is a set $B \subseteq S$ of size ω_1 such that i_s has the same value for all $s \in B$. Show that B induces a cofinal branch through T. $\square$