FORCING EXERCISES 3

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The exercises given here can be found, often along with solutions, in the literature. However, I strongly recommend that instead of looking into the literature you try to figure out the solutions on your own, using the given hints, or by asking for more hints. This way you learn much more.

None of the exercises/results in this series of exercises is due to me, unless explicitly stated. If you find out that some credits are missing or are inaccurate please notify me and I will make corrections.

The dependence in the exercises is vertical, that is, each exercise only uses results discussed in the preceding exercises. (At least I tried to list them in this order.

So far we considered the following properties of forcing posets: chain conditions, Knaster property, closure property, and briefly distributivity. In the following we focus on the connections between the closure property and distributivity, and we also introduce an intermediate property, called strategic closure.

Recall that a forcing poset $\mathbb{P}$ is separative if and only if for every $p, q \in \mathbb{P}$ such that $q \not\leq p$ there is some $r \leq q$ such that $r \perp p$. To motivate this notion it is helpful to consider the situation in Boolean algebras. If $\mathbb{P}$ happens to come from a Boolean algebra, that is, $\mathbb{P} = \mathbb{B} - \{0\}$ for some Boolean algebra $\mathbb{B}$ then $q \not\leq p$ is equivalent to $q - p \neq 0$.

Exercise 3.1 Assume $\mathbb{P}$ is a separative poset. Prove that the following are equivalent for $p, q \in \mathbb{P}$.

(a) $p \leq q$.
(b) $p \Vdash q \in \dot{G}$ where $\dot{G}$ is the canonical name for the generic filter on $\mathbb{P}$.

Exercise 3.2. Assume $\mathbb{P}$ is a forcing poset and κ is a regular cardinal. Prove that (a),(b) are equivalent and each of them implies (c).

(a) The intersection of $< \kappa$ many open dense sets in $\mathbb{P}$ is dense.
(b) If $\langle A_\alpha \mid \alpha < \gamma \rangle$ is a system of maximal antichains in $\mathbb{P}$ where $\gamma < \kappa$ then below any $p \in \mathbb{P}$ there is some $p' \in \mathbb{P}$ such that for every $\alpha < \gamma$ there is some $a \in A_\alpha$ such that $p' \leq a$.
(c) $\mathbb{P}$ is κ-distributive

Also prove that if $\mathbb{P}$ is separative then (a),(b),(c) are all equivalent.

Hint. For (c)$\Rightarrow$(a): Given a family of maximal antichains $\langle A_\xi \mid \xi < \gamma \rangle$ in $\mathbb{P}$ and a generic filter G for $\mathbb{P}$ over $\mathbf{V}$, notice that the function $f : \gamma \to \mathbb{P}$ defined by $f(\xi) =$ the unique element in $G \cap A_\xi$ is in $\mathbf{V}$.

Exercise 3.3. Let $S \subseteq \omega_1$ be a stationary set and let $\mathbb{P}_S$ be the poset defined as follows.

- Conditions are closed countable sets $c \subseteq S$. That is, each $c \in \mathbb{P}_S$ has a largest element which we denote by α_c.
- Ordering is by end-extension. That is, $c \leq d$ if and only if $d = c \cap (\alpha_d + 1)$.

Prove the following facts.

(a) $\mathbb{P}_S$ is ω_1-distributive. (So $\mathbb{P}_S$ does not add any reals, and in particular does not collapse ω_1.)
(b) If G is generic for $\mathbb{P}_S$ over $\mathbf{V}$ then $C_G = \bigcup G$ is a closed unbounded subset of ω_1, and $C_G \subset S$. Thus, if S is co-stationary in the sense of the ground model, S will become an element of the club filter in the sense of the generic extension.
(c) $\mathbf{V}[G] \models \mathsf{CH}$.

Hint. For (a): Let $\langle D_n \mid n \in \omega \rangle$ be a sequence of open dense sets in $\mathbb{P}_S$. Let $c \in \mathbb{P}_S$. Fix a large regular θ and a countable elementary $X \subseteq H_\theta$ such that $\alpha = X \cap \omega_1 \in S$ and all objects of interest are in X. Then using elementarity inductively pick conditions $c_n \in D_n$ such that $c_{n+1} \leq c_n$, and α_{c_n} converge to α.

For (c): Using a density argument, try to encode each real in $\mathbf{V}$ into the generic sequence C_G.

Given a regular cardinal θ, a sequence of sets $\langle x_\alpha \mid \alpha < \theta \rangle$ is called **coherent** if and only if

(a) $x_\alpha \subseteq \alpha$ is unbounded in α for all $\alpha < \theta$, and
(b) if $\bar{\alpha} \in \lim(x_\alpha)$ then $x_{\bar{\alpha}} = x_\alpha \cap \bar{\alpha}$.

Obviously, from the definition of a coherent sequence one sees that limit ordinal are essential for what is going on, so we typically let $x_{\alpha+1} = \{\alpha\}$ and will not worry about what happens at successor ordinals. We will typically consider coherent sequences of the form $\langle c_\alpha \mid \alpha < \theta \rangle$ where each c_α is a closed unbounded subset of α. We will call such a sequence a **coherent sequence of clubs**. A **thread** through such a coherent sequence is a closed unbounded set $C \subseteq \theta$ such that $C \cap \alpha = c_\alpha$ for every $\alpha \in \lim(C)$. Notice that limiting to coherent sequences of clubs is not a loss of generality. We could define a thread through a coherent sequence to be an unbounded set $X \subseteq \theta$ such that $X \cap \alpha = x_\alpha$ for each $\alpha \in \lim(X)$; by taking topological closures of X and x_α's we get a coherent sequence of clubs and a thread through it. Notice also that the thread for a coherent sequence of clubs is unique, i.e. if C, C' are both threads through a coherent sequence of clubs $\langle c_\alpha \mid \alpha < \theta \rangle$ then $C = C'$. Notice also that letting $c_\alpha = \alpha$ we obtain a coherent sequence of clubs with thread θ, so the notion of a coherent sequence becomes interesting only if we put some additional condition on it that causes some tension.

Given an infinite cardinal κ, a coherent sequence of clubs $\langle c_\alpha \mid \alpha < \kappa^+ \rangle$ is a $\square_\kappa$-sequence if and only if $\mathsf{otp}(c_\alpha) \leq \kappa$ for all $\alpha < \kappa^+$. Notice that only limit ordinals in the interval (κ, κ^+) are relevant here. We say that $\square_\kappa$ holds iff there is a $\square_\kappa$-sequence.

Given a regular cardinal θ we say that $\langle c_\alpha \mid \alpha < \theta \rangle$ is a $\square_{<\theta}$-sequence if and only if $\langle c_\alpha \mid \alpha < \theta \rangle$ is a coherent sequence of clubs without thread. We say that $\square(\theta)$ holds if and only if there is a $\square(\theta)$-sequence. Notice that if κ is an infinite cardinal then any $\square_\kappa$-sequence is a $\square(\kappa^+)$-sequence.

Exercise 3.? Let κ be an infinite cardinal. Prove the following.

(a) $\square_\omega$ holds.

(b) If $\langle c_\alpha \mid \alpha < \kappa^+ \rangle$ is a $\square_\kappa$-sequence and κ is regular then for every limit $\alpha < \kappa^+$ we have $\mathsf{otp}(c_\alpha) \leq \kappa$, and $\mathsf{otp}(c_\alpha) = \kappa$ if and only if $\mathsf{cf}(\alpha) = \kappa$. So if κ is singular then we always have $\mathsf{otp}(c_\alpha) < \kappa$.

(c) Assume $\langle c_\alpha \mid \alpha < \kappa^+ \rangle$ is a coherent sequence such that $\mathsf{otp}(c_\alpha) < \alpha$ for all $\alpha < \kappa^+$. Prove that there is a $\square_\kappa$-sequence.

TO BE CONTINUTED...