FORCING EXERCISES 4

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The exercises given here can be found, often along with solutions, in the literature. However, I strongly recommend that instead of looking into the literature you try to figure out the solutions on your own, using the given hints, or by asking for more hints. This way you learn much more.

None of the exercises/results in this series of exercises is due to me, unless explicitly stated. If you find out that some credits are missing or are inaccurate please notify me and I will make corrections.

The dependence in the exercises is vertical, that is, each exercise only uses results discussed in the preceding exercises. (At least I tried to list them in this order.

In the following exercises we will inverstigate preservation of stationarity under combinatorial properties of forcing posets.

Exercise 4.1. Assume κ is regular and $\mathbb{P}$ is a κ-c.c. poset. Prove the following.

(a) $p \Vdash$ "$\dot{C}$ is a club in κ" then there is some club C in κ such that C is in the ground model and $p \Vdash \check{C} \subseteq \dot{C}$. Thus, any club $C \subseteq \kappa$ in a generic extension contains some club from the ground model.

(b) If $S \subseteq \kappa$ is a stationary set in the ground model then S remain stationary in the generic extension.

Hint. For (a): Consider $C = \{\xi < \kappa \mid p \Vdash \check{\xi} \in \dot{C}\}$. To see that this works, notice that by the chain condition, for every $\alpha < \kappa$ there are $< \kappa$ possible candidates for the least element of $\dot{C}^G$ above α, where G runs through all generic filters for $\mathbb{P}$.

Exercise 4.2. Assume κ is regular and $\mathbb{P}$ is a κ-closed poset. If $S \subseteq \kappa$ is stationary in the sense of the ground model then S is stationary in the sense of any generic extension via $\mathbb{P}$.

More generally, that the same is true for any strategically $(\kappa+1)$-closed poset.

Hint. For a contradiciton, assume there is some name $\dot{C}$ and some condition $p \in \mathbb{P}$ such that $p \Vdash$ "$\dot{C}$ is a club in κ and $\check{S} \cap \dot{C} = \varnothing$". Using the closure, inductively keep fixing initial segments of $\dot{C}$ and construct a club $C \subseteq \kappa$ in the ground model that is disjoint from S.

Exercise 4.3. Let $\mathbb{P}$ be the following poset.

- Conditions are closed bounded subset of ω_1. So every condition $c \in \mathbb{P}$ has a largest element, which we denote by α_c.
- Ordering is by end-extension. So $c \le d$ if and only if $d = c \cap (\alpha_d + 1)$.

Notice that $\mathbb{P} = \mathbb{P}_{\omega_1}$ where $\mathbb{P}_S$ is the poset from Exercise 3.3. Prove the following.

(a) $\mathbb{P}$ is ω_1-closed.

(b) If $S \subseteq \omega_1$ is in the club filter then $\mathbb{P}_S$ has a dense subset that is ω_1-closed. (But $\mathbb{P}_S$ itself may not be ω_1-closed.)

(c) Letting $C_G = \bigcup G$ where G is generic for $\mathbb{P}$ over $\mathbf{V}$, the set $C_G \in \mathbf{V}[G]$ is a closed unbounded subset of ω_1 that does not contain any closed unboudned $C \subseteq \omega_1$ such that $C \in \mathbf{V}$.

(d) If $S \subseteq \omega_1$ is stationary-co-stationary in the ground model then $\mathbb{P}_S$ is not ω_1-closed.

Hint. For (c) use a density argument.

For the following exercise recall that given regular cardinals $\gamma < \kappa$ we let

$$S^\kappa_\gamma = \{\xi < \kappa \mid \mathsf{cf}(\xi) = \gamma\}.$$

Exercise 4.4. Assume $\gamma < \kappa$ are regular and $\mu^{<\gamma} < \kappa$ for all $\mu < \kappa$. If $S \subseteq S^\kappa_\gamma$ is stationary in the sense of ground model then its stationarity is preserved by any γ^+-closed forcing.

Note: The new element in this exercise is that we consider stationary sets concentrating on cofinalities much smaller than the cardinal κ whose stationary subsets are considered. T allows to weaken the closure property imposed on $\mathbb{P}$.

Hint. Let $S \subseteq S^\kappa_\gamma$ be stationary, $\dot{C}$ be a $\mathbb{P}$-name and $p \in \mathbb{P}$ be a condition such that $p \Vdash$ "$\dot{C}$ is a club in κ". Let θ be large regular. Using the elementary chain construction build an elementary $X \prec H_\theta$ such that

- all object of interest are in X,
- $\alpha = X \cap \kappa \in S$, and
- ${}^{<\gamma}X \subseteq X$.

Fix a strictly increasing sequence $\langle \alpha_\xi \mid \xi < \gamma \rangle$ converging to α. Notice that each proper initial segment of this sequence is in X. Using the elementarity of X and the closure properties of X and $\mathbb{P}$ construct a decreasing sequence of conditions $p_\xi \in X$ below p along with a sequence of ordinals β_ξ that are forced by p_ξ to be elements of $\dot{C}$ and such that $\alpha_\xi \leq \beta_\xi < \alpha$.

TO BE CONTINUTED....