REVIEW OF ORDINAL DEFINABILITY FOR THE CLASS MATH 285A

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First recall the following from the basic course in logic. There is a Σ_1-definable (over $\mathbf{V}$) relation $\mathbf{Sat}(\mathcal{L}, M, \varphi, s)$ such that if $\mathcal{L}$ is a first-order language, $M \in \mathbf{V}$ is a structure for $\mathcal{L}$, φ is an $\mathcal{L}$-formula and s is an evaluation of variables in M then

$$\mathbf{Sat}(\mathcal{L}, M, \varphi, s) \text{ iff } M \models_{\mathcal{L}} \varphi[s].$$

We will deal with language $\mathcal{L}$ of set theory and its extensions obtained by adding constant symbols and predicate symbols. Denote the language of set theory by LST. Let $\Phi(u, v, w, z)$ be the LST-formula defining the relation $\mathbf{Sat}$.

Let $A \subseteq \mathbf{V}$; here A is a collection that may not necessarily be a class, that is, A may not be definable by an LST-formula. But we have a unary predicate symbol $\dot{A}$ denoting A. We say that a set $x \in \mathbf{V}$ is ordinal definable from elements of A if and only if there is an LST-formula $\varphi(w, u_1, \ldots, u_n, v_1, \ldots, v_m)$, ordinals $\alpha_1, \ldots, \alpha_n$, and $a_1, \ldots, a_m \in A$ such that

$$a \text{ is the unique } y \in \mathbf{V} \text{ such that } \varphi[y, \alpha_1, \ldots, \alpha_n, a_1, \ldots, a_n]. \tag{1}$$

Denote the collection of all sets x that are ordinal definable from elements of A by $\mathbf{OD}_A$.

From now on assume that A is a class; so either A is a set or we have an LST-formula that defines A in $\mathbf{V}$, possibly from parameters in $\mathbf{V}$.

Make sure you understand the following facts.

Fact 1. The formula φ in (1) can be replaced with an LST-formula.

Fact 2. x is ordinal definable from A and its elements if and only if there is an LST-formula $\varphi(w, u_1, \ldots, u_n, v_1, \ldots, v_m)$, an ordinal γ, ordinals $\alpha_1, \ldots, \alpha_n < \gamma$, and elements $a_1, \ldots, a_m \in A \cap V_\gamma$ such that x is the unique $y \in V_\gamma$ satisfying

$$V_\gamma \models \varphi[y, \alpha_1, \ldots, \alpha_n, a_1, \ldots, a_m]. \tag{2}$$

Hint. This relies on the Reflection Theorem.

Fact 3. $\mathbf{OD}_A$ is a class. Moreover, if A has a definition in the parameter $b \in \mathbf{V}$ then $\mathbf{OD}_A$ has a definition in the parameter b.

Define $\mathbf{HOD}_A$ as follows.

$$x \in \mathbf{HOD}_A \iff \mathsf{trcl}(\{x\}) \subseteq \mathbf{OD}_A \tag{3}$$

Fact 4. $\mathbf{HOD}_A$ is a transitive class and $\mathbf{On} \subseteq \mathbf{HOD}_A$. Moreover, if A is transitive then $A \subseteq \mathbf{HOD}_A$.

Fact 5. $HOD_A \models \mathsf{ZF}$.

Hint. Recall that by the inner model criterion (review in Math281A, Fall 2010), it is sufficient to verify that $\mathbf{HOD}_A$ satisfies the following: (a) is a transitive class containing all ordinals, (b) satisfies the Σ_0-separation, and (c) the following variant of covering: If $X \in \mathbf{V}$ is such that $X \subseteq \mathbf{HOD}_A$ then there is $Y \in \mathbf{HOD}_A$ such that $X \subseteq Y$.

Fact 6. Assume $<_A$ is a well-ordering of the class A that is definable from parameters $b_1, \dots, b_k$ in $\mathbf{V}$. Define $<$ on $\mathbf{HOD}_A$ by letting $x < y$ iff

$$(\gamma^x, \varphi^x, \alpha_1^x, \dots, \alpha_n^x, a_1^x \dots, a_m^x) <_{\mathrm{lex}} (\gamma^y, \varphi^y, \alpha_1^y, \dots, \alpha_n^y, a_1^y \dots, a_m^y)$$

where $(\gamma^x, \varphi^x, \alpha_1^x, \dots, \alpha_n^x, a_1^x \dots, a_m^x)$ is the lexicographically least tuple

$$(\gamma, \varphi, \alpha_1, \dots, \alpha_n, a_1 \dots, a_m)$$

as in (2), and similarly for y.

Then $<$ is a well-ordering on $\mathbf{OD}_A$ which is ordinal definable from parameters b_i and the parameter b which occurs in the definition of class A.

Fact 7. If A is transitive and has a well-ordering that is ordinal definable from elementso of A then $\mathbf{HOD}_A \models \mathsf{ZFC}$. In particular, letting $\mathbf{HOD} = \mathbf{HOD}_\varnothing$, we have $\mathbf{HOD} \models \mathsf{ZFC}$.