MATH 285A Spring 2011: HOMEWORK 1

2012-04-09

Exercise 1.1. Let $\mathbb{P}$ be a poset, $p \in \mathbb{P}$, τ be a $\mathbb{P}$-name and σ be a $\mathbb{P}$-name such that $p \Vdash \tau \in \sigma$.

Prove that there is a $\mathbb{P}$-name τ' such that

(a) $p \Vdash \tau = \tau'$, and
(b) for every $\langle r, \rho \rangle \in \tau'$ the name ρ is in $\mathrm{dom}(\pi)$ for some $\pi \in \mathrm{dom}(\sigma)$.

Conclude that $\mathsf{rank}(\tau') \leq \mathsf{rank}(\mathbb{P}) + \mathsf{rank}(\sigma)$.

Hint. The argument resembles that in the proof of the maximality principle. Notice that since $\mathbb{P}$ is a set there are not too many possibilities for names that "are forced to be elements" of σ.

Exercise 1.2. Let $\mathbb{P}$ be a poset and $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a poset. Use Exercise 1.1. to find suitable α such that such that letting

$$\mathbb{P} *_\alpha \dot{\mathbb{Q}} = \{(p, \dot{q}) \in \mathbb{P} \times V_\alpha \mid \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}\}$$

is the correct definition of $\mathbb{P} * \dot{\mathbb{Q}}$.

Also observe that if $\beta > \alpha$ then

$$\mathsf{id} : \mathbb{P} *_\alpha \dot{\mathbb{Q}} \to \mathbb{P} *_\beta \dot{\mathbb{Q}}$$

is an embedding such that for every $(p, \dot{q}) \in \mathbb{P} *_\beta \dot{\mathbb{Q}}$ there is $(\bar{p}, \dot{\bar{q}}) \in \mathbb{P} *_\alpha \dot{\mathbb{Q}}$ such that $(\bar{p}, \dot{\bar{q}}) \equiv (p, \dot{q})$ where "$\equiv$" denotes the conjunction "$\leq$ & $\geq$". So in particular, the identity above is a dense embedding.

Exercise 1.3. Given a complete embedding $\pi : \mathbb{P} \to \mathbb{Q}$ work out all details of the standard facts concerning the transfer of names and generics. For instance, verify the following:

(a) If G is a generic for $\mathbb{Q}$ over $\mathbf{V}$ then $\bar{G} = \pi^{-1}[G]$ is generic for $\mathbb{P}$ over $\mathbf{V}$.
(b) Letting $\pi^*(\sigma) = \{(\pi(r), \pi^*(\rho)) \mid (r, \rho) \in \sigma\}$ where σ is a $\mathbb{P}$-name, we have $\sigma^{\bar{G}} = \pi^*(\sigma)^G$.
(c) For any $p \in \mathbb{P}$, any Σ_0-formula $\varphi(v)$ and any $\mathbb{P}$-name σ we have

$$p \Vdash_{\mathbb{P}} \varphi(\sigma) \iff \pi(p) \Vdash_{\mathbb{Q}} \varphi(\pi^*(\sigma))$$

(d) If φ is a Σ_1-formula then we still have $\Longrightarrow$ in (c).
(e) Prove that if π is a dense embedding then π is a complete embedding, and additionally if G is generic for $\mathbb{P}$ over $\mathbf{V}$ then $G' = \pi[G] \uparrow$ (i.e. G' is the upward closure of $\pi[G]$ in $\mathbb{Q}$) is generic for $\mathbb{Q}$ over $\mathbf{V}$.
(f) Given a poset $\mathbb{P}$ prove that $i : \mathbb{P} \to \mathsf{RO}(\mathbb{P})$ defined by $i(p) = (\overline{p\downarrow})^\circ$ is a dense embedding.

Exercise 1.4. Let $\mathbb{P}$, $\mathbb{Q}$ be posets and $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a poset. Verify that the maps

$$\begin{aligned} i_{\mathbb{P}} : \mathbb{P} \to \mathbb{P} \times \mathbb{Q} \quad & \text{defined by} \quad p \mapsto (p, 1_{\mathbb{Q}}) \\ i_{\mathbb{Q}} : \mathbb{P} \to \mathbb{P} \times \mathbb{Q} \quad & \text{defined by} \quad p \mapsto (1_{\mathbb{P}}, q) \\ i : \mathbb{P} \to \mathbb{P} * \dot{\mathbb{Q}} \quad & \text{defined by} \quad p \mapsto (p, \dot{1}_{\mathbb{Q}}) \end{aligned}$$

are complete embeddings.

Exercise 1.5. Let $\pi : \mathbb{P} \to \mathbb{Q}$ be an embedding. Prove the equivalence of the following.

(a) π is a complete embedding, i.e. by our definition $\pi[A]$ is a maximal antichain in $\mathbb{Q}$ whenever A is a maximal antichain in $\mathbb{P}$.
(b) $\pi[D]$ is a predense set in $\mathbb{Q}$ whenever Q is a predense set in $\mathbb{P}$.
(c) Every $q \in \mathbb{Q}$ has a reduct in $\mathbb{P}$, that is an element $p \in \mathbb{P}$ such that $\pi(p') \parallel q$ whenever $p' \leq p$.

Exercise 1.6. I sketched the proof of the fact that if G is generic for $\mathbb{P}$ over $\mathbf{V}$ and H is generic for $\dot{\mathbb{Q}}^G$ over $\mathbf{V}[G]$ then

$$K = \{(p, \dot{q}) \mid p \in G \;\&\; \dot{q}^G \in H\}$$

is generic for $\mathbb{P} * \dot{\mathbb{Q}}$ over $\mathbf{V}$. So given a dense $D \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ such that $D \in \mathbf{V}$ we need to see that $D \cap K \neq \varnothing$. This boiled down to showing that D^G is dense in $\dot{\mathbb{Q}}^G$, which in turn relied on the fact that given $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ the set

$$\bar{D} = \{p' \in \mathbb{P} \mid (p', \dot{q}') \in D \;\&\; p' \Vdash_{\mathbb{P}} \dot{q}' \leq \dot{q}\}$$

is dense in $\mathbb{P}$. Fill in the details in the proof here and complete the proof of the fact.

Exercise 1.7. Prove the following fact mentioned in the lecture. Assuming $\mathbb{P}$, $\mathbb{Q}$ are posets and σ is a $\mathbb{Q}$-name such that

(a) σ^H is a generic filter for $\mathbb{P}$ over $\mathbf{V}$ whenever H is generic for $\mathbb{Q}$ over $\mathbf{V}$, and
(b) for every $p \in \mathbb{P}$ there is some H as in (a) such that $p \in \sigma^H$.

Then the map

$$p \mapsto \| \check{p} \in \sigma \|_{\mathsf{RO}(\mathbb{Q})}$$

is a complete embedding of $\mathbb{P}$ into $\mathsf{RO}(\mathbb{Q})$.