# MATH 285A Spring 2011: HOMEWORK 2

**2012-04-12**

**Exercise 2.1. Regular open algebras.** Given a topological space $(X, \mathcal{T})$ recall that a set $A \subseteq X$ is regular open if and only if $\bar{A}^\circ = A$. Prove the following.

(a) For every $A \subseteq X$ the set $\bar{A}^\circ = A$ is regular open.
(b) If $A \subseteq X$ is open then $\bar{A}^\circ$ is the smallest regular open subset of $X$ containing $A$ as a subset.
(c) Letting $\mathsf{RO}(X)$ be the collection of all regular open subsets of $X$, the poset $(\mathsf{RO}(X), \subseteq)$ is a complete Boolean algebra with the following operations:

$$\bigvee \mathcal{A} = \left(\overline{\bigcup \mathcal{A}}\right)^\circ$$
$$\bigwedge \mathcal{A} = \left(\bigcap \mathcal{A}\right)^\circ$$

Notice that you have to check the properties of finite algebraic operations, in particular the existence of complements and the distributivity laws which are most demanding.

**Exercise 2.2.** Let $\mathbb{A}, \mathbb{B}$ be complete Boolean algebras and $\pi : \mathbb{A} \to \mathbb{B}$ be a Boolean algebra homomorphism, that is, $\pi$ preserves finite algebraic operations. Recall that $\mathbb{A}^+ = \mathbb{A} - \{\mathbf{0}\}$ and similarly for $\mathbb{B}$. Prove that the following are equivalent.

(a) $\pi$ is an injective complete embedding of $\mathbb{A}$ into $\mathbb{B}$; here "complete" means that $\pi$ preserves arbitrary infinite joins and meets.
(b) The map $\pi$ maps $\mathbb{A}^+$ into $\mathbb{B}^+$, and $\pi \restriction \mathbb{A}^+ : \mathbb{A}^+ \to \mathbb{B}^+$ is a regular embedding in the sense of forcing.

**Exercise 2.3** Let $\pi : \mathbb{P} \to \mathbb{R}$ be a regular embededing and $\dot{Q}$ be a name for the quotiont $\mathbb{R}/\pi[G]$. Let $q \in \mathbb{R}$ and $p \in \mathbb{P}$ be such that $p \Vdash_{\mathbb{P}} \check{q} \in \dot{Q}$. Prove that $p$ is a reduct of $q$.

**Exercise 2.4.** Let $\pi_1 : \mathbb{P} \to \mathbb{P} * \dot{\mathbb{Q}}$ be the regular embedding given by $\pi_1(p, \dot{q}) = (p, \dot{1}_{\dot{\mathbb{Q}}})$, and let $G$ be a generic filter for $\mathbb{P}$ over $\mathbf{V}$.

(a) Determine the quotient $\mathbb{P} * \dot{\mathbb{Q}}/\pi_1[G]$ and justify that it makes sense to write $\mathbb{P} * \dot{\mathbb{Q}}/G$.
(b) Prove that $\mathbb{P} * \dot{\mathbb{Q}}/G$ is forcing equivalent to $\dot{\mathbb{Q}}^G$ in the sense of the model $\mathbf{V}[G]$.

**Hint.** For (b), try to densely embed $\mathbb{P} * \dot{\mathbb{Q}}/G$ into $\dot{\mathbb{Q}}^G$.

**Exercise 2.5.** Let $\pi : \mathbb{P} \to \mathbb{R}$ be a regular embedding between two posets. Let $H$ be a filter generic for $\mathbb{R}$ over $\mathbf{V}$, let $G = \pi^{-1}[H]$, and let $\mathbb{Q} = \mathbb{R}/\pi[G]$. Prove that $H$ is generic for $\mathbb{Q}$ over $\mathbf{V}[G]$.

**Hint.** Given a dense $D \subseteq \mathbb{Q}$ such that $D \in \mathbf{V}[G]$, show that $D \cap H \neq \varnothing$. Let $D = \dot{D}^G$ where $\dot{D}$ is a $\mathbb{P}$-name and let $p \in G$ be such that

$$p \Vdash_{\mathbb{P}} \text{“}\dot{D} \text{ is a dense subset of } \dot{\mathbb{Q}}.\text{”}$$

. Prove that the set

$$D' = \{r \in \mathbb{R} \mid (\exists p' \leq_{\mathbb{P}} p)(\exists d \in \mathbb{R})(r \leq_{\mathbb{R}} \pi(p'), d \ \&\ p' \Vdash_{\mathbb{P}} \text{“}\check{d} \in \dot{D}\text{”}\}$$

is dense below $\pi(p)$ in $\mathbb{R}$.

**Exercise 2.6.** Let $\pi : \mathbb{Q} \to \mathbb{P}$ be an order-preserving map. Prove the basic facts from the lecture:

(a) $\pi$ is a projection iff for every $q \in \mathbb{Q}$ and every $p \leq \pi(q)$ there is some $q'$ compatible with $q$ such that $\pi(q') \leq p$.
(b) $\pi$ is a projection iff for every $q \in \mathbb{Q}$ we have $\pi[q \downarrow]$ is dense below $\pi(q)$.
(c) If $\pi$ is a projection then $\pi^{-1}[D]$ is dense in $\mathbb{Q}$ whenever $D$ is an open dense subset of $\mathbb{P}$.
(d) If $\pi$ is a projection and $H$ is generic for $\mathbb{Q}$ over $\mathbf{V}$ then $\pi[H] \uparrow$ is generic for $\mathbb{P}$ over $\mathbf{V}$.
(e) If $\pi$ is a projection then there is a regular embedding of $\mathbb{P}$ into $\mathsf{RO}(\mathbb{Q})$.

**Exercise 2.7.** Verify that the following maps are projections.

(a) $\pi_1 : \mathbb{P} \times \mathbb{Q} \to \mathbb{P}$ given by $\pi_1(p, q) = p$.
(b) $\pi_2 : \mathbb{P} \times \mathbb{Q} \to \mathbb{Q}$ given by $\pi_2(p, q) = q$.
(c) $\pi : \mathbb{P} * \dot{\mathbb{Q}} \to \mathbb{P}$ given by $\pi(p, \dot{q}) = p$.

**Exercise 2.8.** Given a projection $\pi : \mathbb{Q} \to \mathbb{P}$ let $\pi^* : \mathbb{P}$-names $\to \mathbb{Q}$-names be defined as follows.

$$\pi^*(\sigma) = \{(r, \pi^*(\rho) \mid (\exists p \in \mathbb{P})(\pi(r) \leq p \;\&\; (p, \rho) \in \sigma)\}.$$

Prove that if $H$ is generic for $\mathbb{Q}$ over $\mathbf{V}$ and $G = \pi[H] \uparrow$ then $\pi^*(\sigma)^G = \sigma^H$. Also show that if $\pi$ is surjective then the same conclusion holds if we let

$$\pi^*(\sigma) = \{(r, \pi^*(\rho)) \mid (\pi(r), \rho) \in \sigma\}.$$

**Exercise 2.9.** Let $\pi : \mathbb{P} \to \mathbb{R}$ be a regular embedding. Prove that the map $\sigma : \mathbb{P} \to \mathsf{RO}(\mathbb{P})$ defined by

$$\sigma(r) = ||\check{r} \in \dot{\mathbb{Q}}||_{\mathsf{RO}(\mathbb{P})}$$

is a projection; here $\dot{Q}$ is a name for the quotient $\mathbb{R}/\pi[G]$.