MATH 285A Spring 2011: HOMEWORK 3

2012-04-12

Exercise 3.1. Let $\pi : \mathbb{R} \to \mathbb{P}$ be a projection, let H be generic for $\mathbb{R}$ over $\mathbf{V}$ and let $G = \pi[H]\uparrow$. Finally let $\mathbb{Q}$ be the quotient $\mathbb{R}/G$. Prove that H is generic for $\mathbb{Q}$ over $\mathbf{V}[G]$.

Hint. Let $D = \dot{D}^G$ be an open dense subset of $\mathbb{Q}$. Show that $D \cap H$ is nonepty. For this, fix $p \in G$ such that $p \Vdash_{\mathbb{P}}$ "$\dot{D}$ is open dense in $\dot{\mathbb{Q}}$" and pick $r \in \mathbb{R}$ such that $\pi(r) \leq p$. Then show that $D^* = \{d \in \mathbb{R} \mid \pi(d) \Vdash_{\mathbb{P}} \check{d} \in \dot{D}\}$ is dense below r in $\mathbb{R}$.

Exercise 3.2. (Maximality principle for the uniqueness existential quantifier.) Assume $\mathbf{V}$ is a model of ZF, that is, we work without the Axiom of Choice. Let $\mathbb{P}$ be a poset, let $p \in \mathbb{P}$, let $\varphi(v)$ be a formula and assume $p \Vdash_{\mathbb{P}} (\exists! v)\varphi(v)$. Prove that there is a $\mathbb{P}$-name σ such that $p \Vdash \varphi(\sigma)$. (This is the Choice-less variant of the maximality principle Monroe pointed out during the lecture.)

Hint. Show that there is an ordinal α such that for every $p \in \mathbb{P}$ there is some name $\sigma' \in V_\alpha$ with $p \Vdash \varphi(\sigma')$. Then try to build a name σ so that $\mathrm{dom}(\sigma) \subseteq V_\alpha$. The name will thus be pretty big, as you cannot use maximal antichains nor pick representatives. Instead, use the fact that if $p_1 \Vdash \varphi(\sigma_1)$ and $p_2 \Vdash \varphi(\sigma_2)$ then $\sigma_1^G = \sigma_2^G$ whenever $p_1, p_1 \in G$.

Exercise 3.3. Let $\mathbb{P}$ be a poset, $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is a poset", and let G be generic for $\mathbb{P}$ over $\mathbf{V}$. Let $(p, \dot{q}), (p', \dot{q}')$ be conditions in $\mathbb{P} * \dot{\mathbb{Q}}$ such that $p, p' \in G$ and $\dot{q}^G, \dot{q}'^G$ are compatible in $\dot{\mathbb{Q}}^G$. Prove that $(p, \dot{q}), (p', \dot{q}')$ are compatible in $\mathbb{P} * \dot{\mathbb{Q}}$.

Equivalently, prove that if $\bar{p} \leq p, p'$ then $\bar{p} \Vdash_{\mathbb{P}}$ "$\dot{q}, \dot{q}'$ are incompatible in $\dot{\mathbb{Q}}$".

Exercise 3.4. Recall that if $\mathbb{P} = (P, \leq)$ is a poset then the quasiordering $\leq^*$ on P is defined by

$$p \leq q \iff p \Vdash \check{q} \in \dot{G}$$

where $\dot{G}$ is a name for a generic filter for $\mathbb{P}$. Equivalently, we have

$$p \leq^* q \iff \bar{p} \,||\, q \text{ whenever } \bar{p} \leq p.$$

(a) Prove that $\leq^*$ is indeed a quasiordering and the relation $\equiv^*$ defined by

$$p \equiv^* q \implies p \leq^* q \;\&\; q \leq^* p$$

is an equivalence relation on P extending $\leq$, that is, $\leq \subseteq \leq^*$. So $\mathbb{P}^* = (P, \leq^*)$ is also a forcing poset. The poset $\mathbb{P}^*$ is called the separative quotient of $\mathbb{P}$.

(b) Show tht $\mathbb{P}^*$ is separative, i.e. the quasiordering $\leq^*$ on P is separative.

(c) Show that the identity map $\mathsf{id} : \mathbb{P} \to \mathbb{P}^*$ is a dense embedding. Notice that the quotient $\mathbb{P}^* / \equiv^*$ is then a separative poset, and the quotient map $\mathsf{id}/\equiv^*$ is a dense embedding. So forcing with $\mathbb{P}$ is equivalent to forcing with the separative poset $\mathbb{P}^*$.

(d) Show that any dense embedding $\pi : \mathbb{P} \to \bar{\mathbb{P}}$ is a projection.

Exercise 3.5. Let $\mathbb{P}$ be a poset, $\dot{Q}$ be a $\mathbb{P}$-name for a poset, and κ be a regular cardial. Prove the implication

$$\Big(\varphi(\mathbb{P}, \kappa) \;\&\; \Vdash_{\mathbb{P}} (\dot{Q}, \kappa)\Big) \implies \varphi(\mathbb{P} * \dot{\mathbb{Q}}, \kappa)$$

for the following $\varphi(u, v)$.

(a) $\varphi(\mathbb{P}, \kappa) \equiv \mathbb{P}$ is κ-distributive.
(b) $\varphi(\mathbb{P}, \kappa) \equiv \mathbb{P}$ is κ-closed.
(c) $\varphi(\mathbb{P}, \kappa) \equiv \mathbb{P}$ is κ-directed closed.
(d) $\varphi(\mathbb{P}, \kappa) \equiv \mathbb{P}$ is κ-Knaster.
(e) $\varphi(\mathbb{P}, \kappa) \equiv \mathbb{P}$ is separative.

Exercise 3.6. Let $\mathbb{B}$ be a Boolean algebra. Prove the following.

(a) If $a \in \mathbb{B}$, $Z \subseteq \mathbb{B}$ and $\bigwedge Z$ exists then so does $\bigwedge_{z \in Z}(a \vee z)$ and
$$\bigwedge_{z \in Z}(a \vee z) = a \vee \bigwedge Z.$$
Dual statement holds for $\bigvee$.

(b) If $Z \subseteq \mathbb{B}$ let $Z^c = \{z' \mid z \in Z\}$ where z' is the complement of z in $\mathbb{B}$. Now assume $\mathbb{B}$ is a complete Boolean algebra. Prove the infinite DeMorgan Laws:
$$(\bigwedge Z)^c = \bigvee Z^c \quad \text{and} \quad (\bigvee Z)^c = \bigwedge Z^c.$$

(c) Given a complete Boolean algebra $\mathbb{B}$, prove that if $Z \subseteq \mathbb{B}$ is a set closed under complements then so is
$$Z' = \{\bigwedge X \mid X \subseteq Z\} \cup \{\bigvee X \mid X \subseteq Z\}.$$

Hint. For (a) you will need the finite the finite distributive laws for Boolean algebras.