ELEMENTARY FACTS ON ULTRAPOWERS FOR THE CLASS MATH 285A

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You should understand the facts on this sheet and work through their proofs. I will give hints; the proofs are otherwise straightforward and based on the hints and your knowledge so far you should be able to figure out the details.

We will consider ultrapower maps $\pi : M \to N$ where M is a (set-size or class-size) model of a sufficient fragment of ZFC. For simplicity we will consider either the full ZFC or ZFC^-, that is we drop the power set axiom. Of course, both M, N are transitive.

Obviously if $\pi : M \to N$ is any elementary map then $\mathrm{rng}(\pi)$ is an elementary substructure of N. The following important observation says that non-trivial embeddings must move ordinals.

Fact 1. Let $N \models \mathsf{ZFC}$ and X be an elementary substructure of N such that $\mathbf{On} \cap N \subseteq X$. Then $X = N$.

Hint. The point is that N is a model of choice, so for any $x \in N$ there is some ordinal $\alpha \in N$ and some bijection $f : \alpha \to x$ such that $f \in N$.

Let $Z \in M$. We will consider an ultrafilter U over Z that measures all sets in $\mathcal{P}(Z) \cap M$. That is, for every $A \subseteq Z$ such that $A \in M$ either $A \in U$ or $Z - A \in U$. However, we do not require that $U \in M$. The case where $M = \mathbf{V}$ and $U \in \mathbf{V}$ we considered so far is an instance of this situation, but there are other important situations where we cannot have $U \in M$. In the situation described above we will say that U is an M-ultrafilter.

Given M, U as above, the ultrapower construction still makes sense if we restrict the class of functions used in the ultrapower construction as follows.

- The domain of the ultrapower is determined by all functions $f : Z \to M$ such that $f \in M$.

Thus, letting

$$D = \{f \mid f : Z \to M \ \&\ f \in M\}$$

we define $\sim_U$ on D as usual, that is, we let $f \sim_U g$ iff $\{z \in Z \mid f(z) = g(z)\} \in U$. Check that this makes sense, $\sim_U$ is an equivalence relation on D, and that we can then run the usual ultrapower construction and define the interpretation E_U on $D/\sim_U$ of the $\dot{\in}$ symbol by letting $[f]_U\, E_U [g]_U$ iff $\{z \in Z \mid f(z) \in g(z)\} \in U$. The treatment of proper classes here is the same as in the treatment of ultrapowers we saw before. Denote the outcome of the ultrapower construction by $\mathbb{D}(M, U)$. Also check that the Łoś theorem holds, that is, given a formula $\varphi(v_1, \dots, v_\ell)$ and functions $f_1, \dots, f_\ell \in D$ we have

$$\mathbb{D}(M, U) \models \varphi([f_1]_U, \dots, [f_\ell]_U) \iff \{z \in Z \mid M \models \varphi(f_1(z), \dots, f_\ell(z))\} \in U.$$

It follows from the Łoś theorem that the map $\pi : M \to \mathbb{D}(M, U)$ defined by $\pi(a) = [c_a]_U$ where c_a is the constant function with value a is elementary. We call this map the ultrapower map/embedding. Check that the following hold.

$$[f]_U = \pi(f)([\mathsf{id}]_U) \text{ for every } f \in D \tag{1}$$

and

$$U = \{A \in \mathcal{P}(Z) \cap M \mid [\mathsf{id}]_U \in \pi(A)\}. \tag{2}$$

Hint. Use the Łoś theorem.

In what follows we will assume that $\mathbb{D}(M, U)$ is well-founded, and write $\mathsf{Ult}(M, U)$ to indicate this. Also, we identify the ultrapower map π with the map from M into $\mathsf{Ult}(M, U)$ which is technically the composition of the collapsing map with π. Given an elementary map $\pi : M \to N$ where M, N are transitive, a set $Z \in M$ and an element $i^* \in \pi(Z)$ let $U = U_{\pi,i^*}$ be defined as follows

$$U = \{A \in \mathcal{P}(Z) \cap M \mid i^* \in \pi(A)\}$$

Check that the following hold.

(a) U is an M-ultrafilter over Z.
(b) U is principal iff $i^* \in \operatorname{rng}(\pi)$.

U is called the M-ultrafilter derived from π, i^*.

We will need a way to see when π is the ultrapower map associated with $\mathsf{Ult}(M, U)$. The following fact gives a complete answer. Before stating the fact, recall that a set $a \in N$ is definable in N from parameters $a_1, \ldots, a_\ell$ iff there is a formula $\varphi(v, u_1, \ldots, v_\ell)$ which defines it, i.e. such that a is the unique object x in N satisfying $N \models \varphi(x, a_1, \ldots, a_\ell)$. For $n \in \omega$ we way that a is Σ_n-definable in N from parameters $a_1, \ldots, a_\ell$ if there is such a Σ_n-formula. The Σ_n-hull of $B \subseteq N$ in N is the class of all $a \in N$ that are Σ_n-definable in N from parameters in B.

Fact 2. Let $\pi : M \to N$ be elementary where M, N are transitive ZFC^--models. Let $Z \in M$ and $i^* \in \pi(Z)$. Let finally U be the M-ultrafilter derived from π, i^*. The following are equivalent.

(a) $N = \mathsf{Ult}(M, U)$ and π is the associated ultrapower embedding.
(b) Every $a \in N$ is of the form $\pi(f)(i^*)$ for some $f \in M$ such that $\operatorname{dom}(f) = Z$.
(c) N is the Σ_0-hull of $\operatorname{rng}(\pi) \cup \{i^*\}$ in N.

If any of (a) – (c) holds then $i^* = [\mathsf{id}]_U$.

Hint. For (a)⇒(b), assuming $N = \mathsf{Ult}(M, U)$ define $\sigma : N \to N$ by

$$\sigma([f]_U) = \pi(f)(i^*). \tag{3}$$

Show σ is elementary and $\sigma([\mathsf{id}]_U) = i^*$. Observe that $\sigma = \mathsf{id}$.

(b)⇒(c) is easy.

For (c)⇒(b), if $a \in N$, $a_1, \ldots, a_\ell \in M$ and $\varphi(u, v_1, \ldots, v_\ell)$ is a Σ_0-formula that defines a in N from $\pi(a_1), \ldots, \pi(a_\ell)$ let $f : Z \to M$ be the function defined as follows.

$$f(z) = \begin{cases} \text{the unique } y \in M \text{ such that } M \models \varphi(y, a_1, \ldots, a_\ell) \text{ if such a } y \text{ exists} \\ 0 \text{ otherwise} \end{cases}$$

Show that f does the job.

For (b)⇒(a), define $\sigma : \mathbb{D}(M,U) \to N$ as in (3) and show it is elementary. Conclude that $\mathbb{D}(M,U)$ is well-founded, and view σ as a map $\sigma : \mathsf{Ult}(M,U) \to N$. Show that σ is surjective.

Next we will need to know the behaviour of ultrapower maps at ordinals as well as the the relationship between the ultrapower, the original model, and $\mathbf{V}$. Below are some useful special cases. Recall that the critical point $\mathsf{cr}(\pi)$ of an elementary embedding $\pi : M \to N$ is the first ordinal moved by π.

Fact 3. Let $\pi : M \to N$ be elementary and $\kappa = \mathsf{cr}(\pi)$. The following hold.

(a) κ is regular in M.
(b) π is discontinuous at all ordinals α such that $\mathsf{cf}^M(\alpha) = \kappa$.
(c) π is continuous at all ordinals α such that $\mathsf{cf}^M(\alpha) < \kappa$.
(d) If $V^M_\kappa = V_\kappa$ then $V^N_\kappa = V_\kappa$, and if $V^M_{\kappa+1} = V_{\kappa+1}$ then $V^N_{\kappa+1} = V_{\kappa+1}$.
(e) If $V^M_{\kappa+1} = V^N_{\kappa+1}$, or more generally if $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap N$ then κ is strongly inaccessible, Mahlo,... in the sense of M.

Hint. For (a) pick a cofinal strictly increasing $g : \gamma \to \kappa$ such that $g \in M$ and $\alpha < \kappa$, and look at $\pi(g)$. Similarly proceed in (b) and (c). For (d) do the induction on rank for the first part and then look how subsets of V_κ are mapped by π. For (e), assume there is an injection $f : \kappa \to V_\alpha$ where $\alpha < \kappa$ and look at $\pi(f)$.

Now recall that an ultrafilter U over Z is γ-complete iff for any $\bar{\gamma} < \gamma$ and any family $\mathcal{A} = \{A_\xi \mid \xi < \bar{\gamma}\}$ of sets in U the intersection $\bigcap \mathcal{A}$ is in U. For M-ultrafilters we define γ-completeness similarly with the only difference that we restrict ourself to families $\mathcal{A} \in M$. The completeness of U is the largest κ such that U is κ-complete, and analogously for M-ultrafilters. (Check that such κ exists!)

Fact 4. If $\pi : M \to N$ is an elementary embedding with $\kappa = \mathsf{cr}(\pi)$ and i^* is such that U_{π,i^*} is nonprincipal then the completeness of U_{π,i^*} is at least κ.

If π is the ultrapower map by U then the completeness of U is precisely κ.

Conversely, if U is an M-ultrafilter with completeness κ and $\pi : M \to N$ is the associated ultrapower embedding the $\mathsf{cr}(\pi) = \kappa$.

Hint. Concerning the second conclusion, notice that there is some $f : Z \to \kappa$ such that $\pi(f)(i^*) = \kappa$. Letting $A_\alpha = \{z \in Z \mid f(z) < \alpha\}$, the family $\{A_\alpha \mid \alpha < \kappa\}$ witnesses the completeness is at most κ.

In most applications it will always be the case that the completeness of the derived ultrafilter is equal to the critical point; in each case this will follow from the situation. But for the moment I do not see how to prove that the completeness is equal to the critical point in the case π is not the ultrapower embedding.

We now summarize properties of ultrapowers in two important special cases.

The first case is when $Z = \kappa$ and U is a κ-complete ultrafilter on κ. Recall that U is normal iff any regressive $f : A \to \kappa$ defined on some $A \in U$ is constant on a set in U. For M-ultrafilters we only require this for $f \in M$.

Fact 5. Let $\pi : M \to N$ be an elementary embedding with critical point κ. Let $U = U_{\pi,\kappa}$ be the M-ultrafilter on $Z = \kappa$ derived from π. Then the completeness of U is κ and U is normal.

Fact 6. Let U be a κ-complete ultrafilter on $Z = \kappa$. Let $\pi : M \to N$ be the associated ultrapower map. The following hold.

(a) If U is normal then $\kappa = [\mathsf{id}]_U$.
(b) For each ordinal α, the map π is continuous at α iff $\mathsf{cf}^M(\alpha) \neq \kappa$.
(c) If $M = \mathbf{V}$ then $V^N_{\kappa+1} = V_{\kappa+1}$, $\pi \restriction V_\kappa = \mathsf{id} \restriction V_\kappa$, and ${}^\kappa N \subseteq N$.
(d) If $M = \mathbf{V}$ and λ is a strong limit cardinal of cofinality distinct from κ then $\pi(\lambda) = \lambda$, and $\pi(\lambda^{+\alpha}) = \lambda^{+\alpha}$ for all $\alpha < \kappa$.
(e) For each regular $\mu \neq \kappa$ there is a μ-closed proper class of fixpoints of π.
(f) $\kappa^{+N} \geq \kappa^+$, so if $M = \mathbf{V}$ then $\kappa^{+N} = \kappa^+$.
(g) $\pi(\kappa) < (2^\kappa)^+$.
(h) $U \notin N$.

Hint. (a) is straightforwared, and (b) and (c) follows from Fact 3 (b) – (d). For (d) compute the size of $\pi(\bar{\lambda})$ for $\bar{\lambda} <]lambda$ and use (b). (e) follows from (d) and (f) from (c). (g) follows from direct calculation. Finally (g) follows from the calculation for (g) and from the fact that $\pi(\kappa)$ is completely determined by U and $V_{\kappa+1}$.

The second case of interest is where $\kappa \leq \lambda$ are cardinals, κ is regular, and U is a κ-complete ultrafilter over

$$\mathcal{P}_\kappa(\lambda) = \{z \in \mathcal{P}(\lambda) \mid \mathsf{card}(z) < \kappa \ \&\ z \cap \kappa \in \kappa\}.$$

We say that

- U is **fine** iff $C_\alpha = \{z \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in z\}$ is an element of U for each $\alpha < \lambda$.
- U is **normal** iff every regressive function $f : A \to \lambda$ where $A \in U$ is constant on some set in U. Recall that f is regressive iff $f(z) \in z$ for all $z \in A$.

A normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$ is also called λ-supercompact ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Note that the case $\lambda = \kappa$ can be reduced to the previous case where we let $Z = \kappa$, so the notion of λ-supercompact ultrafilter is new only for $\lambda > \kappa$.

Fact 7. Let $\pi : \mathbf{V} \to N$ be an elementary embedding with critical point κ and let $\lambda \geq \kappa$ be a cardinal such that the following are satisfied.

(i) $\pi(\kappa) > \lambda$.
(ii) ${}^\lambda N \subseteq N$.

Define $U \subseteq \mathcal{P}(\mathcal{P}_\kappa(\lambda))$ by letting

$$A \in U \iff \pi[\lambda] \in \pi(A)$$

Then U is a normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$ whose completeness is κ.

Hint. By Fact 3 (e) κ is strongly inaccessible. By (ii) $\pi[\lambda] \in N$, and hence by (i) $\pi[\lambda] \in \pi(\mathcal{P}_\kappa(\lambda))$. By Fact 4 the completeness of U is at least κ. On the other hand, given $\alpha < \kappa$ the set

$$A_\alpha = \{z \in \mathcal{P}_\kappa(\lambda) \mid \mathsf{otp}(z) > \alpha\} \in U$$

hence the family $\{A_\alpha \mid \alpha < \kappa\}$ witnesses the completeness of U is at most κ. The verifiation that U is fine is easy. To see that U is normal let $f : A \to \lambda$ be regressive where $A \in U$ and look at $\pi(f)(\pi[\lambda])$.

Remark. Notice that the proof of κ-completeness in Fact 7 is superfluous, as any normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$ is automatically κ-complete. So when talking about fine normal measures on $\mathcal{P}_\kappa(\lambda)$ we may omit saying "κ-complete" explicitly.

Hint. If $\gamma < \kappa$ then $C_\gamma \in U$ and $\gamma \subseteq z$ for each $z \in C_\gamma$.

Fact 8. Let U be fine normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$ and $\pi : \mathbf{V} \to N$ be the ultrapower embedding by U. The following hold.

(a) $\mathsf{cr}(\pi) = \kappa$.
(b) For any $\alpha \leq \lambda$ the ordinal α is represented in the ultrapower by the function $f_\alpha : \mathcal{P}_\kappa(\lambda) \to \kappa$ where $f_\alpha(z) = \mathsf{otp}(\alpha \cap z)$.
(c) In particular, κ is represented by the function $f_\kappa(z) = \kappa \cap z$ and λ is represented by the function $f_\lambda(z) = \mathsf{otp}(z)$.
(d) $\pi(\kappa) > \lambda$.
(e) $\pi[\lambda] = [\mathsf{id}]_U$.
(f) ${}^\lambda N \subseteq N$.
(g) π is discontinuous at every ordinal α such that $\kappa \leq \mathsf{cf}(\alpha) \leq \lambda$ and continuous at all ordinals α of cofinality outside of this interval.
(h) $\pi(\mu) = \mu$ for every strong limit cardinal $\mu > \lambda$ whose cofinality is outside of the interval $[\kappa, \lambda]$.
(i) $\mathcal{P}(\lambda) \subseteq N$. More generally, $\mathcal{P}(\mathcal{P}_\kappa(\lambda)) \subseteq N$.
(j) $\alpha^{+N} = \alpha^+$ for all $\alpha \leq \lambda$.
(k) $\pi(\kappa) < (\kappa^{\lambda^\kappa})$.
(l) $U \notin N$.

Hint. (b) is proved by induction on α using the normality of U. (c) follows from (b) and (d) from (c). "$\subseteq$ in (e)" is proved using the fineness of U, and "$\supseteq$" using the normality. To see (f) first observe that from (e) we have $\pi \restriction \lambda \in N$. Now if $\langle x_\xi \mid \xi < \lambda \rangle$ is such that $x_\xi \in N$ for each $\xi < \lambda$ then $x_\xi = \pi(f_\xi)([\mathsf{id}]_U)$ for some $f_\xi : \mathcal{P}_\kappa(\lambda) \to \mathbf{V}$. It is now enough to observe that $\langle \pi(f_\xi) \mid \xi < \lambda \rangle \in N$. The continuity in (g) follows similarly as in Fact 6. The discontinuity follows from the fact that $\pi[\lambda] \in N$. (h) follows by calculations similar to those in the proof of Fact 6. To see the first conclusion in (i), notice that $A = (\pi \restriction \lambda)^{-1}[\pi(A)]$ whenever $A \subseteq \lambda$. Alternatively, A is represented in the ultrapower by the function $f_A(z) = h_z[A \cap z]$ where $h : z \to \mathbf{V}$ is the Mostowski collapsing isomorphism. Similarly, the second conslusion in (i) follows from the fact that $\pi \restriction \mathcal{P}_\kappa(\lambda) \in N$ and $A = (\pi \restriction \mathcal{P}_\kappa(\lambda))^{-1}[\pi(A)]$ whenever $A \subseteq \mathcal{P}_\kappa(\lambda)$. Similarly as before the function $f_A(z) = h_z[A \cap z]$ represents A in the ultrapower. (j) follows from (i) and (k),(l) follow by considerations similar to those in the proof of Fact 6.