DIAMOND, GCH AND WEAK SQUARE

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Abstract. Shelah proved recently that if \( \kappa > \omega \) and \( S \subseteq \kappa^+ \) is a stationary set of ordinals of cofinality different from \( \text{cf}(\kappa) \) then \( 2^\kappa = \kappa^+ \) implies \( \diamondsuit(S) \). We show that for singular \( \kappa \), an elaboration on his argument allows to derive \( \diamondsuit(T) \) from \( 2^\kappa = \kappa^+ + \square^* \) where \( T = \{ \delta < \kappa^+ | \text{cf}(\delta) = \text{cf}(\kappa) \} \). This gives a strong restriction on the existence of saturated ideals on \( \kappa^+ \).

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It is a well-known fact that \( \diamondsuit \) implies \( 2^{< \lambda} = \lambda \). In many situations the converse is also true. Jensen [3] proved that CH does not imply \( \diamondsuit \), so when looking for the converse one has to focus on \( \lambda > \omega_1 \). Let
\[
S_\mu^\lambda = \{ \delta < \lambda | \text{cf}(\delta) = \mu \}
\]
and
\[
T_\kappa = S_{\text{cf}(\kappa)}^{\kappa^+}.
\]
Gregory observed that GCH below \( \omega_2 \) implies \( \diamondsuit_{\omega_2}(S_\omega^{\omega_2}) \). A sequence of improvements on his result, mainly by Gregory [4], Jensen (unpublished) and Shelah [7], resulted in the following theorem whose proof can be found in [2].

**Theorem 0.1** (Gregory, Jensen, Shelah). If \( 2^{< \kappa} = \kappa \) and \( 2^\kappa = \kappa^+ \) then \( \diamondsuit_\kappa^+(S) \) holds whenever \( S \subseteq \kappa^+ \) is a stationary set of ordinals of cofinality different from \( \text{cf}(\kappa) \). If \( \kappa \) is singular and additionally \( \square_\kappa \) holds then \( \diamondsuit_\kappa^+(T_\kappa) \).

Shelah also proved that for regular \( \kappa \), the condition \( 2^\kappa = \kappa^+ + \square_\kappa \) is not sufficient to guarantee \( \diamondsuit_\kappa^+(T_\kappa) \), so the absolute ZFC result is possible only for singular \( \kappa \).

The question remained whether the localized GCH, i.e. the equality \( 2^\kappa = \kappa^+ \) alone implies \( \diamondsuit_\kappa^+ \). Shelah proved this to be true for sufficiently large \( \kappa \), and recently [8] found an argument that proves it for every uncountable cardinal \( \kappa \); see Komjáth’s paper [6] for a simplified proof and an elaboration on Shelah’s argument.

**Theorem 0.2** (Shelah). Let \( \kappa > \omega \) and \( 2^\kappa = \kappa^+ \). Then \( \diamondsuit_\kappa^+(S) \) holds for every stationary \( S \subseteq \kappa^+ \) that is disjoint with \( T_\kappa \).

This note combines arguments from the proof of Theorem 0.1 with Shelah’s argument for Theorem 0.2 to give a proof of \( \diamondsuit_\kappa^+(S) \) for \( S \subseteq T_\kappa \).

**Theorem 0.3** (Main Theorem). Assume \( \kappa \) is a singular cardinal and \( T \subseteq T_\kappa \) is stationary with stationarily many reflection points. Then
\[
2^\kappa = \kappa^+ + \square_\kappa^* \implies \diamondsuit_\kappa^+(T).
\]

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It was proved by Cummings, Foreman and Magidor [1] that for singular \( \kappa \), the principle \( \square^*_\kappa \) is consistent with the requirement that every stationary \( T \subseteq T_\kappa \) has stationarily many reflection points. Consequently, in their model we have \( \dot{\mathcal{Q}}_{\kappa^+}(S) \) for all stationary \( S \subseteq \kappa^+ \).

**Corollary 0.4.** Assume \( \kappa \) is a singular cardinal. Then

\[
2^\kappa = \kappa^+ + \square^*_\kappa \implies \dot{\mathcal{Q}}_\kappa(T_\kappa).
\]

Both the above theorem and its corollary provides a very strong restriction on the existence of saturated ideals on \( \kappa^+ \) and provide a close link between the study of such ideals and the PCF-theory.

Rinot recently extended the result of this paper to the situation where weak square is replaced by a variant of approachability property and also showed that, relatively to the existence to a supercompact cardinal, it is consistent that \( \square^*_\kappa \) fails but \( \dot{\mathcal{Q}}_{\kappa^+}(S) \) holds for every stationary \( S \subseteq \kappa^+ \).

1. The Argument

We begin with splitting Shelah’s argument into two steps. We first isolate a combinatorial statement that alone implies the existence of a \( \mathcal{Q}_\lambda(S) \)-sequence in ZFC; we denote this statement by \( \mathcal{Q}_\lambda(S) \). This statement is implicit in the arguments in Shelah [8] and Komjáth [6]. It turns out that the implication \( \mathcal{Q}_\lambda(S) \implies \dot{\mathcal{Q}}_\lambda(S) \) is true no matter whether \( \lambda \) is a successor cardinal or not. The second step is a proof that \( \mathcal{Q}_\lambda(S) \) holds, which relies on the localized GCH if \( \lambda = \kappa^+ \) and \( S \) concentrates on points of cofinality distinct from \( \text{cf}(\kappa) \) which gives the original Shelah’s result, and on the weak square if \( \kappa \) is singular and \( S \) concentrates on points of cofinality \( \text{cf}(\kappa) \) which gives the result in Theorem 0.3. Our approach owes a lot to Komjáth’s exposition in [6].

**Definition 1.1.** Let \( \lambda \) be a regular cardinal and \( S \subseteq \lambda \). We say that the pair \( \langle x_\xi | \xi < \lambda \rangle, \langle A_\delta | \delta \in S \rangle \) witnesses \( \mathcal{Q}_\lambda(S) \) iff the following three conditions are met.

(a) \( \langle x_\xi | \xi < \lambda \rangle \) is an enumeration of \( [\lambda]^{<\lambda} \).

(b) \( A_\delta \subseteq \delta \) and \( \text{card}(A_\delta) < \text{card}(\delta) \) whenever \( \delta \in S \).

(c) For every \( Z \subseteq \lambda \) there is a stationary \( S' \subseteq S \) such that for every \( \delta \in S' \) there are uncountably many \( \alpha < \delta \) for which there is \( \beta < \delta \) satisfying \( \alpha, \beta \in A_\delta \) and \( Z \cap \alpha = x_\beta \).

We say that \( \mathcal{Q}_\lambda(S) \) holds iff there are \( \langle x_\xi \rangle_\xi \) and \( \langle A_\delta \rangle_\delta \) as above.

Notice that \( \mathcal{Q}_\lambda(S) \) postulates the existence of an enumeration of \( [\lambda]^{<\lambda} \) of length \( \lambda \), so it imposes some constraints on the behaviour of the exponential function below \( \lambda \). In particular, if \( \lambda = \kappa^+ \) then \( \mathcal{Q}_\lambda(S) \) implies \( 2^\kappa = \kappa^+ \). Notice also that (b) in the above definition stipulates that the cardinality of \( A_\delta \) is strictly smaller than that of \( \delta \), which together with (c) implies that without loss of generality \( S \) can be viewed as a set of singular ordinals. Of course, this has a non-trivial meaning only when \( \lambda \) is inaccessible. Finally observe that if there is a pair \( \langle x_\xi | \xi < \lambda \rangle, \langle A_\delta | \delta \in S \rangle \) witnessing \( \mathcal{Q}_\lambda(S) \) then for every enumeration \( \langle x'_\xi | \xi < \lambda \rangle \) there is a sequence \( \langle A'_\delta | \delta \in S \rangle \) such that the pair \( \langle x'_\xi \rangle_\xi, \langle A'_\delta \rangle_\delta \) witnesses \( \mathcal{Q}_\lambda(S) \). To see this, pick any \( f : \lambda \to \lambda \) such that \( x_\beta = x'_{f(\beta)} \) for all \( \beta < \lambda \) and let \( A'_\delta = A_\delta \cup f[A_\delta] \) for all \( \delta \in S \) satisfying \( f[\delta] \subseteq \delta \).

**Lemma 1.2.** Let \( \lambda \) be a regular cardinal, \( S \subseteq \lambda \) and \( \mathcal{Q}_\lambda(S) \) hold. Then there is a pair \( \langle x_\xi | \xi < \lambda \rangle, \langle A_\delta | \delta \in S \rangle \) satisfying the following.
(a) $\langle x_\xi \mid \xi < \lambda \rangle$ is an enumeration of $[\lambda \times \lambda]^{<\lambda}$.

(b) $A_S \subseteq \delta$ and $\text{card}(A_S) < \text{card}(\delta)$ whenever $\delta \in S$.

(c) For every $Z \subseteq \lambda \times \lambda$ there is a stationary $S' \subseteq S$ such that for every $\delta \in S'$ there are unboundedly many $\alpha < \delta$ for which there is $\beta < \delta$ satisfying $\alpha, \beta \in A_S$ and $Z \cap (\alpha \times \alpha) = \{ x_\delta \cap (\alpha \times \alpha) \}$.

**Proof.** Pick a pair $\langle \langle \xi \rangle \mid \xi < \lambda \rangle$, $\langle B_\delta \mid \delta \in S \rangle$ witnessing $\bigcirc_\lambda(S)$. Let $f : \lambda \times \lambda \to \lambda$ be a bijection and $C_f = \{ \delta < \lambda \mid f[\delta \times \delta] = \delta \}$. To each $\delta \in S$ pick $C_\delta$ to be a subset of $\text{lim}(C_f) \cap \delta$ of size strictly smaller than $\text{card}(\delta)$ that is cofinal in $\delta$ if such a set exists; let $C_\delta = \emptyset$ otherwise. Letting $x_\delta = f^{-1}[y_\beta]$ and $A_S = B_\delta \cup C_\delta$, we obtain a pair $\langle x_\xi \mid \xi < \lambda \rangle$, $\langle A_S \mid \delta \in S \rangle$ as in the conclusion of the lemma. To see this, it suffices to verify clause (c) in the statement of the lemma.

Given any $Z \subseteq \lambda \times \lambda$, let $S' \subseteq S$ be the stationary set obtained by applying $\bigcirc_\lambda(S)$ to $f[Z]$. Let $\delta \in S' \cap \text{lim}(C_f)$. If $\alpha < \delta$, pick $\alpha \in C_\delta$ such that $\alpha \leq \lambda$. Since $S'$ satisfies (c) in Definition 1.1 with $f[Z], y_\beta$ and $B_\delta$ in place of $Z, x_\beta$ and $A_S$, there are $\alpha', \beta \in B_\delta$ such that $\alpha \leq \alpha'$ and $f[Z] \cap \alpha' = y_\beta$. Then $f[Z] \cap \alpha = y_\beta \cap \alpha$ and the conclusion follows immediately from the fact that $\alpha \in C_f$. \hfill \square

With the statement $\bigcirc_\lambda(S)$ in hand, one can reformulate the first step in Shelah's argument into the following proposition. It reduces the proof of $\bigcirc_\lambda(S)$ to the proof of $\bigcirc_\lambda(S)$ and works even for cardinals $\lambda$ that are not successors, which is slightly more than Shelah has originally proved. The second step in Shelah's argument can be then viewed as a proof of $\bigcirc_{\kappa^+}(S)$ from the localized GCH. We will show how to obtain $\bigcirc_{\kappa^+}(S)$ from the additional assumption that $\square^*_{\kappa}$ holds in situations where localized GCH does not seem to suffice.

**Proposition 1.3.** Let $\lambda$ be regular and $S \subseteq \lambda$ be stationary. Then

$$
\bigcirc_\lambda(S) \implies \bigcirc_\lambda(S)
$$

**Proof.** Let $\langle x_\xi \mid \xi < \lambda \rangle$, $\langle A_S \mid \delta \in S \rangle$ be a pair satisfying the conclusion of Lemma 1.2. For $x \subseteq \lambda \times \lambda$ we write $(x)_\xi$ to denote $\{ \xi < \lambda \mid (x, \xi)_\xi \in x \}$. Consider sequences $(X_\xi, C_\xi \mid \xi < \theta)$ of length $\theta \leq \lambda$ such that $X_\xi \subseteq \lambda$, $C_\xi$ is closed unbounded in $\lambda$ and, letting

$$
V_\xi^S = \{ (\alpha, \beta) \in A_S \times A_S \mid (\forall \eta < \xi)(X_\eta \cap \alpha = (x_\beta)_\eta \cap \alpha) \},
$$

for every $\xi < \theta$ and $\delta \in S \cap C_\xi$ either $\text{dom}(V_\xi^S)$ is bounded in $\delta$ or else $V_\xi^S \supseteq V_{\xi+1}^S$. Notice that the non-strict inclusion $V_\xi^S \supseteq V_{\xi+1}^S$ holds anyway whenever $\xi < \xi^+$. The crucial observation is that any sequence $(X_\xi, C_\xi \mid \xi < \theta)$ as above has length strictly below $\lambda$. Assume for a contradiction that this fails, that is, there is such a sequence with $\theta = \lambda$. Let $S'$ come from the application of Lemma 1.2 to the pair $\langle x_\xi \rangle_{\xi \in \lambda}$ and to set

$$
Z = \{ (\xi, \zeta) \mid \xi \in X_\xi \},
$$

and let $\delta \in S' \cap \Delta\{ C_\xi \mid \xi < \lambda \}$ be such that $\delta > \kappa$ if $\lambda = \kappa^+$ and $\delta$ is a cardinal if $\lambda$ is inaccessible. We have arbitrarily large $\alpha < \delta$ for which there exists $\beta < \delta$ such that $\alpha, \beta \in A_S$ and $Z \cap (\alpha \times \alpha) = x_\beta \cap (\alpha \times \alpha)$, so for each $\xi < \delta$ the set $\text{dom}(V_\xi^S)$ is unbounded in $\delta$. Since $\delta \in S \cap C_\xi$ whenever $\xi < \delta$, from the properties of the sequence $(X_\xi, C_\xi \mid \xi < \theta)$ we obtain $V_\xi^S \supseteq V_{\xi+1}^S$ whenever $\xi < \xi^+ < \delta$. This is a contradiction, as $V_\xi^S \subseteq A_S \times A_S$ and $\text{card}(A_S) < \text{card}(\delta)$. $\blacksquare$
Pick a sequence $\langle X_\xi, C_\xi \mid \xi < \theta \rangle$ as in the previous paragraph which has no proper extension. Then $\theta < \lambda$. Letting

$$D_\delta = \bigcup \{ \{ x_\beta \cap \alpha \mid \langle \alpha, \beta \rangle \in V_\delta \},$$

the sequence $\langle D_\delta \mid \delta \in S \rangle$ is a $\diamondsuit_\lambda (S)$-sequence. To see this, pick an arbitrary $X \subseteq \lambda$ and a closed unbounded $C \subseteq \lambda$. There exists some $\delta \in S \cap C$ such that $\text{dom}(V_\delta)$ is unbounded in $\delta$ and $X \cap \alpha = \{ x_\beta \cap \alpha \mid \langle \alpha, \beta \rangle \in V_\delta \}$, as otherwise we could extend the sequence $\langle X_\xi, C_\xi \mid \xi < \theta \rangle$ by letting $X_\delta = X$ and $C_\delta = C$, in contradiction with its maximality. But then $X \cap \delta = D_\delta$. □

We now focus on proofs of $\Diamond_\lambda (S)$. The point of introducing $\Diamond_\lambda (S)$ is that it is often easier to give a direct proof of $\Diamond_\lambda (S)$ than a direct proof of $\Diamond_\lambda (S)$. This is clear from Shelah’s argument in [8] which in our notation is a proof of $\Diamond_{\kappa^+} (S)$. As Proposition 1.3 also holds for inaccessible $\lambda$, our hope was that Shelah’s argument may be used for proofs of $\Diamond_\lambda (S)$ for inaccessible $\lambda$. It seems, however, that for inaccessible $\lambda$ the proofs of $\Diamond_\lambda (S)$ may require more new ideas. For instance, the proofs of $\Diamond_\lambda (S^\lambda)$ for a Mahlo cardinal $\lambda$ in [5] and [9] can be easily modified to give proofs of $\Diamond_\lambda (S^\lambda)$, but introducing a $\Diamond_\lambda (S^\lambda)$-sequence into the construction does not seem to enable any strengthening of the results or a simplification of the construction in [9]. For inaccessibles $\lambda$ that are not Mahlo it is not clear either whether an argument using $\Diamond_\lambda (S^\lambda)$ may work. It is certainly clear that constructions of a $\Diamond_\lambda (S^\lambda)$-sequence from “below” as in Propositions 1.4 and 1.5 will not work, essentially for the same reason why constructions of $\Diamond_\lambda (S^\lambda)$ from “below” cannot work, as described in [9]. Analogously as in [9], given any fixed $\Diamond_\lambda (S^\lambda)$-witness $\langle x_\beta \mid \beta < \lambda \rangle$, $\langle A_\delta \mid \delta < \lambda \rangle$, there is a $\langle \lambda, \delta \rangle$-distributive forcing that “kills” such a witness. On the other hand, any construction of a $\Diamond_\lambda (S^\lambda)$-witness from “below” would give rise to the same witness in the ground model and in the generic extension.

Let us turn to the proof of $\Diamond_{\kappa^+} (S)$. As already mentioned above, the next proposition can be viewed as the first step in Shelah’s argument. We include it, as it is a starting point for our variation with weak square.

**Proposition 1.4.** Assume $S \subseteq \kappa^+$ is stationary and disjoint from $T_\xi$. Then

$$2^\kappa = \kappa^+ \implies \Diamond_{\kappa^+} (S).$$

**Proof.** Pick an arbitrary enumeration $\langle y_\xi \mid \xi < \kappa^+ \rangle$ of $\kappa^+ \subseteq \kappa$. The existence of such an enumeration is guaranteed by the localized GCH. Let $g : \varepsilon \times \kappa^+ \to \kappa^+$ be a bijection where $\varepsilon = \text{cf} (\kappa)$. For each $\delta \in S$ pick an increasing (with respect to the inclusion) sequence of sets $\langle A^\delta_\xi \mid \xi < \varepsilon \rangle$ such that $A^\delta_\xi < \kappa$ for all $\xi < \varepsilon$ and $\bigcup_{\xi < \varepsilon} A^\delta_\xi = \delta$.

We show that there is an $\iota < \varepsilon$ such that for every $Z \subseteq \kappa^+$ there are stationarily many ordinals $\delta \in S$ satisfying:

For unboundedly many $\alpha < \delta$ there are $\beta < \delta$ such that

$$\alpha, \beta \in A^\delta_\iota \text{ and } Z \cap \alpha = (g^{-1}[y_\beta]).$$

It follows that letting $A_\delta = A^\delta_\iota$ and $x_\beta = (g^{-1}[y_\beta])$, the pair $\langle x_\beta \mid \beta < \kappa^+ \rangle$, $\langle A_\delta \mid \delta \in S \rangle$, witness $\Diamond_{\kappa^+} (S)$. □

Assume for a contradiction there is no $\iota$ as in the previous paragraph. Then for every $\iota < \varepsilon$ there is a set $Z_\iota \subseteq \kappa^+$ such that (1) holds only on a non-stationary

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1See proof of Proposition 1.3 for the notation $(u)_\eta$. 


subset of $S$. Let $Z = \{ (\iota, \xi) \mid \xi \in Z_i \}$ and $Z' = g[Z]$. The set $S'$ consisting of all $\delta \in S$ such that
\begin{itemize}
  \item $g[\xi \times \alpha] = \alpha$ for cofinally many $\alpha < \delta$ and
  \item $(\forall \alpha < \delta)(\exists \beta < \delta)(Z' \cap \alpha = y_\beta)$
\end{itemize}
is stationary in $\kappa^+$. To each $\delta \in S'$ pick a cofinal strictly increasing sequence $\langle \alpha_\eta \mid \eta < \text{cf}(\delta) \rangle$ such that $g[\xi \times \alpha_\eta] = \alpha_\eta$ for each $\eta < \text{cf}(\delta)$, and to each $\eta < \text{cf}(\delta)$ pick $\beta_\eta < \delta$ such that $Z' \cap \alpha_\eta = y_{\beta_\eta}$. This is possible by the above arrangements for elements of $S'$.

If $\delta \in S'$ then there is an $t(\delta) < \varepsilon$ such that $\alpha_\eta, \beta_\eta \in A^{\delta}_{t(\delta)}$ for cofinally many $\eta < \text{cf}(\delta)$. This follows immediately if $\text{cf}(\delta) < \varepsilon$, as the assignment 
$$
\eta \mapsto \text{the least } t \text{ such that } \alpha_\eta, \beta_\eta \in A^\delta_t
$$
cannot be cofinal in $\varepsilon$, so in fact $\alpha_\eta, \beta_\eta \in A^{\delta}_{t(\delta)}$ for all $\eta < \text{cf}(\delta)$. If $\text{cf}(\delta) > \varepsilon$ this follows by the pigeonhole principle, namely the inverse image of some $A^\delta_t$ under this assignment must have size $\text{cf}(\delta)$. Applying the pigeonhole principle to the assignment $\delta \mapsto t(\delta)$, we obtain a stationary $S'' \subseteq S'$ and an $\iota < \varepsilon$ such that $\iota(\delta) = \iota$ for all $\delta \in S''$.

Pick $\delta \in S''$. By the above arrangements, there are cofinally many $\alpha < \delta$ for which there are $\beta < \delta$ such that $\alpha, \beta \in A^\delta_{t(\delta)}$ and $Z' \cap \alpha = y_{\beta}$. Moreover, the ordinals $\alpha$ can be chosen so that $g[\xi \times \alpha] = \alpha$. It follows that
$$
Z \cap (\varepsilon \times \alpha) = g^{-1}[Z' \cap \alpha] = g^{-1}[y_{\beta}],
$$
so $Z_i \cap \alpha = (g^{-1}[y_{\beta}])$, for all $\alpha, \beta$ as above. Since this is true of any $\delta \in S''$ we obtained a contradiction to the fact that $Z_i$ is a countable model to the statement $Z_i$.

The following proposition shows how to apply a standard construction that uses $\square^*_\kappa$ to prove $\circ_{\kappa^+}(T)$.

**Proposition 1.5.** Assume $\kappa$ is singular and $T \subseteq T_{\kappa}$ is a stationary subset of $\kappa^+$ with stationarily many reflection points. Then
$$2^\kappa = \kappa^+ + \square^*_\kappa \implies \circ_{\kappa^+}(T).$$

**Proof.** We elaborate on the argument from the proof of Proposition 1.4. Let $\varepsilon = \text{cf}(\kappa)$. Fix the following objects:

- Sequences $\langle y_\ell \mid \xi < \kappa^+ \rangle, \langle A^\delta_{t(\delta)} \mid t < \varepsilon \rangle$ and a bijection $g : \varepsilon \times \kappa^+ \to \kappa^+$ as in the proof of Proposition 1.4.
- For each ordinal $\delta < \kappa^+$ an injection $h_\delta : \delta \to \kappa$.
- A $\square^*_\kappa$-sequence $\langle \xi_\delta \mid \delta \in \text{lim } \cap(\kappa, \kappa^+) \rangle$. For each $\delta$ fix an enumeration $\langle \xi_\zeta \mid \zeta < \kappa \rangle$ of the set $\xi_\delta$.
- An increasing (with respect to the inclusion) sequence of sets $\langle B_\iota \mid t < \varepsilon \rangle$ such that $|B_\iota| < \kappa$ for each $\iota$ and $\bigcup_{\iota < \varepsilon} B_\iota = \kappa \times \kappa$.

For each $\delta \in \text{lim } \cap(\kappa, \kappa^+)$ and $\zeta < \kappa$ define a function $f^\delta_\zeta : \delta \to \kappa \times \kappa$ and a sequence of sets $\langle A^\delta_{t(\delta)} \mid t < \varepsilon \rangle$ as follows. 
$$f^\delta_\zeta(\xi) = \langle \eta, h_\eta(\xi) \rangle$$
where $\gamma$ is the least element of $\xi^\delta_\zeta$ strictly above $\xi$ and $\eta = \text{otp}(c^\delta_\zeta \cap \gamma) - 1$ and
$$A^\delta_{t(\delta)} = \langle f^\delta_{t(\delta)} \rangle^1[B_\iota].$$

\footnote{Notice that $\text{otp}(c^\delta_\zeta \cap \gamma)$ is a successor ordinal if $\gamma$ is as above.}
Notice that each \( f^\xi = \) is an injection. By our choice of the sets \( B_i \), we then have \( |A_{\xi,i}^\delta| < \kappa \) and \( \bigcup_{\xi < \kappa} A_{\xi,i}^\delta = \delta \). We also have the following coherency property for the sets \( A_{\xi,i}^\delta \): If \( \delta \) is a limit point of \( c^\xi \) then there is an ordinal \( \zeta < \kappa \) such that
\[
A_{\zeta,i}^\delta \cap \delta = A_{\zeta,i}^{\delta^\zeta}.
\]
To see this notice first that if \( \delta \) is a limit point of \( c^\xi \) then there is \( \zeta < \kappa \) such that \( c^\xi \cap \delta = c^\zeta \), and from the definition of \( f^\xi \) we immediately conclude that \( f^\xi = f^{\delta^\zeta} \cap \delta \).

The rest follows immediately from the definition of \( A_{\xi,i}^\delta \).

Fix an increasing sequence \( \langle \kappa_i \mid i < \varepsilon \rangle \) cofinal in \( \kappa \). For each \( \delta \in T \) and \( i < \varepsilon \) set \( A_i^\delta = \bigcup_{\zeta < \kappa} A_{\zeta,i}^\delta \). Notice that \( |A_i^\delta| < \kappa \), as \( |A_{\zeta,i}^\delta| < |B_i| \) for all \( \xi < \kappa \). Following the ideas from the proof of Proposition 1.4 we prove: There is an \( i < \varepsilon \) such that for every \( Z \subseteq \kappa^+ \) there are stationarily many \( \delta \in T \) satisfying:
\[
(3) \quad \alpha, \beta \in A_i^\delta \text{ and } Z \cap \alpha = (g^{-1}[y_\beta])_i.
\]
It follows that letting \( x_\beta = (g^{-1}[y_\beta])_i \), and \( A_3 = A_i^\delta \) where \( i \) is as above, the pair \( \langle x_\beta \mid \beta < \kappa^+ \rangle, \langle A_3 \mid \delta \in T \rangle \) witnesses \( \bigcap_{\kappa^+} (T) \).

Assume for a contradiction that no such \( i \) above exists. As in the proof of Proposition 1.4 pick a counterexample \( Z_i \) for each \( i < \varepsilon \), let \( Z = \{ (i, \xi) \in \varepsilon \times \kappa^+ \mid \xi \in Z_i \} \) and \( Z' = g[Z] \). Let \( C \) be a closed unbounded subset of \( \kappa^+ \). By our assumption on \( T \), there is a reflection point \( \delta' \) of \( T \) such that:
\[\bullet \quad \delta' \text{ is a limit point of } C.\]
\[\bullet \quad g[\varepsilon \times \alpha] = \alpha \text{ for cofinally many } \alpha < \delta'.\]
\[\bullet \quad (\forall \alpha < \delta') (\exists \beta < \delta') (Z' \cap \alpha = y_\beta).\]

As \( \delta' \) is a reflection point of \( T \), necessarily \( \text{cf}(\delta') > \varepsilon \). Pick an increasing sequence \( \langle \alpha_\eta \mid \eta < \text{cf}(\delta') \rangle \) cofinal in \( \delta' \) such that \( g[\varepsilon \times \alpha_\eta] = \alpha_\eta \) for each \( \eta < \text{cf}(\delta') \). To each \( \eta < \text{cf}(\delta') \) assign some \( \beta_\eta < \delta' \) satisfying \( Z' \cap \alpha_\eta = x_{\beta_\eta} \). It is convenient to pick \( \beta_\eta \) to be least possible. Since \( \text{cf}(\delta') > \varepsilon \), using the pigeonhole principle we conclude that there is some \( i' < \varepsilon \) such that \( \alpha_\eta, \beta_\eta \in A_{i',i'}^\delta \) for cofinally many \( \eta < \text{cf}(\delta') \). Let \( \delta \in T \cap C \cap \text{lim}(c^\xi) \) be a limit point of \( \{ \alpha_\eta \mid \alpha_\eta, \beta_\eta \in A_{i',i'}^\delta \} \). Such a \( \delta \) exists by our choice of \( \delta' \) and \( i' \), and by the fact that \( T \cap \delta' \) is stationary in \( \delta' \). Let \( \xi < \kappa \) be such that \( A_{\zeta,i,i}^\delta = A_{\zeta,i,i}^\delta \cap \delta \) and let \( i(\delta) > i' \) be such that \( \kappa_{i(\delta)} > \xi \). The existence of such a \( \xi \) follows from (2). Then \( A_{i,i,i}^\delta \cap \delta \subseteq A_{i(\delta)}^\delta \), as \( B_i \subseteq B_{i(\delta)} \).

The previous paragraph proves that there is a stationary \( T' \subseteq T \) such that for every \( \delta \in T' \) there is an \( i(\delta) < \varepsilon \) such that for cofinally many \( \alpha < \delta \) there are \( \beta < \delta \) such that \( \alpha, \beta \in A_{i(\delta)}^\delta, Z' \cap \alpha = y_\beta \) and \( g[\varepsilon \times \alpha] = \alpha \). The rest of the proof literally follows the proof of Proposition 1.4. We first find a stationary \( T'' \subseteq T' \) on which \( i(\delta) \) stabilizes; let \( i \) be the stabilized value. Then we unfold \( Z' \) and \( y_\beta \) using \( g \) and conclude that for \( \alpha, \beta \) as above we have \( Z_i \cap \alpha = (g^{-1}[y_\beta])_i \). This yields a contradiction with the fact that \( Z_i \) is a counterexample to (3).

\[\square\]

References

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