26.6.2012.

## Exercises on Fine structure and $\square_\kappa$

1. Prove that $B_\tau$ are coherent and closed
2. Prove that acceptable structures satisfy GCH, and $(H_\tau)^M = J_\tau^B$ when $M = (J_\tau^B, D)$
3. If $M$ is acceptable then $R^1_M \subseteq P^1_M$
4. Let $h_M$ be the uniform $\Sigma_1$-Skolem function for $M$ and $X \subseteq M$. Show $h_M(X) = \{h_M(i,x) \mid x \in X\}$ is the smallest $\Sigma_1$-elementary substructure of $M$ that contains $X$.
5. Assume $\rho^1_{B_\tau} = \kappa$ all $\tau$. Show that if $\mathrm{cf}^V(\kappa^{+L}) > \omega$ then $\langle B_\tau \mid \tau \in C \rangle$ does not have a thread in $V$.

## Exercises on Sean's talk

1. If $U$ is an $\omega$-complete UF on $M$ with $\mathrm{cr}(U) = \kappa$ then $\mathrm{cf}^V(\kappa) > \omega$.
2. If $M, \kappa$ as above, $M = J_\alpha^A$ acceptable, $\mathrm{cf}^V(\kappa), \mathrm{cf}^V(\tau) > \omega$ when $\tau = \kappa^{+M}$. Then $U$ is $\omega$-complete
3. Let $U$ be a normal measure on $M$, $\kappa = \mathrm{cr}(U)$. Then $U$ is $\omega$-complete iff For every countable $(\bar M, \bar U)$ and any $\sigma$ s.t. $\sigma : (\bar M, \bar U) \to (M, U)$ there is a $\Sigma_0$-map $\sigma'$ s.t. the diagram commutes:

$$\begin{tikzcd} M & \\ \bar M \arrow[u, "\sigma"] \arrow[r, "\pi_U"'] & \bar M' \arrow[ul, "\sigma'"'] \end{tikzcd}$$

4. If $(M, U)$ is s.t. $U$ is weakly amenable w.r.t. $M$ and

26.6.2012 (2)

~~3~~ 4. (Continued) $U$ is $\omega$-complete then $(M, U)$ is iterable.

5. If $M = \langle J_\kappa, U\rangle$ is $0^\#$, or any iterable structure then the critical points $\langle \kappa_i \mid i \in On\rangle$ of the iteration constitute a nice set of indiscernibles for $L$, i.e. if ~~.....~~ $\varphi(u_1 \ldots u_e, v_1 \ldots v_k)$ be a formula. Then if $\alpha_1 \ldots \alpha_e$ ~~.....~~ and ~~if~~ $i_1 < \ldots < i_k$, $j_1 < \ldots < j_k$ are s.t. $\alpha_1 \ldots \alpha_e < \kappa_{i_1}, \kappa_{j_1}$ then

$$L \models \varphi(\vec{\alpha}, \kappa_{i_1} \ldots \kappa_{i_e}) \iff L \models \varphi(\vec{\alpha}, \kappa_{j_1} \ldots \kappa_{j_e})$$

6. Assume $j : L \to L$ be elementary with $crit(j) = \kappa$. Then $U = \{x \in P(\kappa) \cap L \mid \kappa \in j(x)\}$ is a weakly amenable normal measure on $L$.

5. Continued: If $M = 0^\#$ then $h(\{\kappa_i \mid i \in On\}) = L$. Also: $\langle \kappa_i \mid i \in On\rangle$ is a closed class.

7. Let $M, N$ be iterable mice s.t. $h_M(\omega) = M$ and $h_N(\omega) = N$. Then $M = N$

8. Let $M, N$ be iterable mice. Then $M$ is an iterate of $N$ or vice versa

9. Let $M = (J_\kappa, U)$ be iterable mouse s.t. $\kappa = cr(U)$ is least possible. Then $M = 0^\#$.

10. Assume $0^\#$ exists.
(a) Show that $\kappa_i^{+L}$ is $\omega$-cofinal in $V$, any $i \in \omega$
(b) Show that $\kappa^{+L}$ is $\omega$-cofinal in $V$ for any $L$-cardinal $\kappa$.

26.6.2012 (3)

11. If $0^\#$ exists then every uncountable V-cardinal is among $\iota_i$, so in particular any V-cardinal is inaccessible, weakly compact, ineffable... in L.

12. Let $\sigma : Q \longrightarrow \tilde{Q}$ $\Sigma_0$ cofinal and $\tilde{\sigma} : M \to \tilde{M}$ be obtained by the pseudoultrapower construction where $wfp(\tilde{M})$ is transitive.
Prove

(a) $[a,f] = \tilde{\sigma}(f)(a)$

(b) $wfp(\tilde{M}) \supseteq \tilde{Q}$

(c) $\tilde{\sigma} \upharpoonright Q = \sigma$

(d) $\tilde{\sigma}$ is $\Sigma_0$-preserving and cofinal

13. Prove the Interpolation Lemma for the map $\sigma : M \longrightarrow N$, i.e. show that if $\tau$ is a cardinal in M then letting $\tilde{M} = Ult(M, \sigma \upharpoonright H^M_\tau)$ and $\tilde{\sigma} : M \longrightarrow \tilde{M}$ be the pseudoultrapower ~~contraction~~ map, there is a unique $\Sigma_0$-preserving $\sigma' : \tilde{M} \to N$ s.t. $\sigma' \upharpoonright \tilde{\tau} = id$ and $\sigma = \sigma' \circ \tilde{\sigma}$.

14. Let $\sigma : H \to H_\theta$ s.t. $H^\omega \subseteq H$. Assume $Q \subseteq H$ and $\sigma : Q \to Q'$ $\Sigma_0$ + cofinal.
Prove: if M is an end-extension of Q s.t. $Q = H^M_\tau$ then $Ult(M, \sigma \upharpoonright Q)$ is w.f..
Do this in two steps:

(a) A map $\sigma : Q \xrightarrow[\Sigma_0]{} Q'$ is $\omega$-complete iff for every countable $\tau : \bar{Q} \xrightarrow[\Sigma_0]{} Q'$ ~~and~~ there is $\bar{\pi} : \bar{Q} \xrightarrow[\Sigma_0]{} Q$ s.t. letting $A = \tau^{-1}[rng(\sigma)]$:

$$\tau \upharpoonright A = \sigma \circ \bar{\pi} \upharpoonright A$$

26.6.2012 (4)

(b) Let $\sigma : H \to H_\theta$ be s.t. ${}^{\omega}H \subseteq H$.
Let $Q \in H$ and $Q' = \bigcup \sigma[Q] = \bigcup_{x \in Q} \sigma(x)$

Show that the map $\sigma \upharpoonright Q : Q \to Q'$ is $\omega$-complete.

Hint Enumerate the range of $\tau$ as $\langle y_i \mid i \in \omega \rangle$ and also all $\Sigma_0$-formulae by $\omega$. Then try to get some sequence $\langle z_i \mid i \in \omega \rangle$ that enumerates a countable $\Sigma$-elementary substructure of $Q$ s.t. $\sigma(z_i) = y_i$ whenever $y_i \in \mathrm{rng}(\sigma)$.