Codes from Polynomials over Finite Fields

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Joint Mathematics Meetings 2021

January 7, 2021

I. What is Coding Theory All About?

MAA Invited Paper Session on Coding Theory and Geometry

- **Friday January 8, 2021, 1:00 p.m.-3:50 p.m.**
  MAA Invited Paper Session on Coding Theory and Geometry
  Organizers:
  Nathan Kaplan, University of California Irvine nckaplan@math.uci.edu
  - 1:00 p.m.
    Applications of finite geometries in coding theory.
    Christine A Kelley*, University of Nebraska-Lincoln
    Michelle Haver, University of Nebraska-Lincoln
    (1163-AI-1443)
  - 1:30 p.m.
    Locally Recoverable Codes with Many Recovery Sets from Number Theory and Geometry.
    Beth Malmskog*, Colorado College
    Kathryn Haymaker, Villanova University
    Gretchen Matthews, Virginia Tech
    (1163-AI-1084)
  - 2:00 p.m.
    Locally Correctable Codes and the Sylvester-Gallai theorem.
    Zeev Dvir*, Princeton University
    (1163-AI-928)
  - 2:30 p.m.
    Some recent results on high rate local codes.
    Shubhangi Saraf*, Rutgers University
    (1163-AI-1672)
  - 3:00 p.m.
    Equiangular lines and spectral graph theory.
    Zilin Jiang, MIT
    Jonathan Tidor, MIT
    Yuan Yao, MIT
    Shengtong Zhang, MIT
    Yufei Zhao*, MIT
    (1163-AI-290)
  - 3:30 p.m.
    Toward classifying multipoint codes.
    Gretchen Matthews*, Virginia Tech
    (1163-AI-1177)

Communication over a Noisy Channel

Suppose we want to communicate over a noisy channel.

I will send you a message: 0 or a 1.

- If I send 0, there is a 90% chance you receive 0.
- If I send 1, there is a 90% chance you receive 1.

Idea: Instead of sending 0 or 1, I will send 000 or 111.

◦ If you receive 010, you 'decode' as 000 because it is likelier that I sent 000 and that there was 1 error than it is that I sent 111 and there were 2 errors.

- If I send 0 or 1, there is a 90% chance you receive the correct message.
- If I send 000 or 111, you receive the correct message with probability

$$(.9)^3 + \binom{3}{1}(.9)^2(.1) = .972.$$

There is a cost for this increased reliability– have to send 3 bits instead of 1.

Major Question

How do we efficiently build redundancy into our set of messages so that we can identify and correct errors?

Coding Theory Basics I

Let $\mathbb{F}_q$ be a finite field of size q.

Definition

- *A code over $\mathbb{F}_q$ of length n is a subset $C \subseteq \mathbb{F}_q^n$.*
- *C is a linear code if it is a linear subspace of $\mathbb{F}_q^n$.*
  That is, if $c_1, c_2 \in C$ then $c_1 + c_2 \in C$ and $\alpha c_1 \in C$ for any $\alpha \in \mathbb{F}_q$.
- *For $\substack{x=(x_1,\ldots,x_n) \\ y=(y_1,\ldots,y_n)} \in \mathbb{F}_q^n$, the Hamming distance between x and y is*

$$d(x, y) = \#\{i \mid x_i \neq y_i\}.$$

- *The Hamming weight of x is $\mathrm{wt}(x) = d(x, \mathbf{0}) = \#\{i \mid x_i \neq 0\}$.*

Example

$\{(0,0,0),(1,1,1)\} \subset \mathbb{F}_2^3$ is a 1-dimensional linear code.

$$d((0,0,0),(1,1,1)) = 3.$$

Coding Theory Basics II

Definition

The *minimum distance* *of a code* C *is*

$$d(C) = \min_{\substack{x,y \in C \\ x \neq y}} d(x,y).$$

○ If C is linear, $d(C)$ is the minimum weight of a nonzero $c \in C$.

$$d(x,y) = d(x-y, y-y) = \mathrm{wt}(x-y)$$

○ In a code with minimum distance d, can correct up to $t = \lfloor \frac{d-1}{2} \rfloor$ errors.

Example

$C = \{(0,0,0),(1,1,1)\} \subset \mathbb{F}_2^3$ has $d(C) = 3$.
You can correct up to $t = \lfloor \frac{3-1}{2} \rfloor = 1$ error.

Main Problem in Combinatorial Coding Theory

We want codes $C \subseteq \mathbb{F}_q^n$ of **large size** and **large minimum distance**.

Definition

Let $A_q(n, d)$ be the maximum size of a code $C \subseteq \mathbb{F}_q^n$ that has minimum distance at least d.

Main Problem in Combinatorial Coding Theory:
Compute values of $A_q(n, d)$.

On the Size of Optimal Three-Error-Correcting Binary Codes of Length 16

Patric R. J. Östergård

***Abstract*—Let $A(n, d)$ denote the maximum size of a binary code with length n and minimum distance d. It has been known for decades that $A(16, 7) = A(17, 8) = 36$ or 37, that is, that the size of optimal 3-error-correcting binary codes of length 16 is either 36 or 37. By a recursive classification via subcodes and a clique search in the final stage, it is shown that the size of optimal such codes is 36.**

attaining the lower bound have been constructed in [13], [14] (see also [10, pp. 57,58]) and the upper bound is from [3]. The problem of determining this particular value is also mentioned in [8, Research Problem 7.18]. The main result of this work is that the best known lower bound is the exact value: $A(17, 8) = 36$.

Tables for $A_2(n, d)$

[Östergård, 2011]: $A_2(17, 8) = 36$.

	d=4	d=6	d=8	d=10	d=12	d=14	d=16
6	4	2	1	1	1	1	1
7	8	2	1	1	1	1	1
8	16	2	2	1	1	1	1
9	20	4	2	1	1	1	1
10	40	6	2	2	1	1	1
11	72	12	2	2	1	1	1
12	144	24	4	2	2	1	1
13	256	32	4	2	2	1	1
14	512	64	8	2	2	2	1
15	1024	128	16	4	2	2	1
16	2048	256	32	4	2	2	2
17	2816-3276	258-340	36	6	2	2	2
18	5632-6552	512-673	64	10	4	2	2
19	10496-13104	1024-1237	128	20	4	2	2
20	20480-26168	2048-2279	256	40	6	2	2
21	40960-43688	2560-4096	512	42-47	8	4	2
22	81920-87333	4096-6941	1024	64-84	12	4	2
23	163840-172361	8192-13674	2048	80-150	24	4	2
24	327680-344308	16384-24106	4096	136-268	48	6	4
25	2^{19}-599184	17920-47538	4096-5421	192-466	52-55	8	4
26	2^{20}-1198368	32768-84260	4104-9275	384-836	64-96	14	4
27	2^{21}-2396736	65536-157285	8192-17099	512-1585	128-169	28	6
28	2^{22}-4792950	131072-291269	16384-32151	1024-2817	178-288	56	8

Figure: Brouwer's tables of upper and lower bounds for $A_2(n, d)$

What is $A_2(17, 6)$?

Bounds on the minimum distance of linear codes over GF(2)

length:	$1 \le n \le 256$
dimension:	$1 \le k \le 256$
Constructions for marked entries are missing	

n/k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1															
2	2	1														
3	3	2	1													
4	4	2	2	1												
5	5	3	2	2	1											
6	6	4	3	2	2	1										
7	7	4	4	3	2	2	1									
8	8	5	4	4	2	2	2	1								
9	9	6	4	4	3	2	2	2	1							
10	10	6	5	4	4	3	2	2	2	1						
n/k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
11	11	7	6	5	4	4	3	2	2	2	1					
12	12	8	6	6	4	4	4	3	2	2	2	1				
13	13	8	7	6	5	4	4	4	3	2	2	2	1			
14	14	9	8	7	6	5	4	4	4	3	2	2	2	1		
15	15	10	8	8	7	6	5	4	4	4	3	2	2	2	1	
16	16	10	8	8	8	6	6	5	4	4	4	2	2	2	2	1
17	17	11	9	8	8	7	6	6	5	4	4	3	2	2	2	2
18	18	12	10	8	8	8	7	6	6	4	4	4	3	2	2	2
19	19	12	10	9	8	8	8	7	6	5	4	4	4	3	2	2
20	20	13	11	10	9	8	8	8	7	6	5	4	4	4	3	2
n/k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
21	21	14	12	10	10	8	8	8	8	7	6	5	4	4	4	3
22	22	14	12	11	10	9	8	8	8	8	7	6	5	4	4	4
23	23	15	12	12	11	10	9	8	8	8	8	7	6	5	4	4
24	24	16	13	12	12	10	10	8	8	8	8	8	6	6	4	4
25	25	16	14	12	12	11	10	9	8	8	8	8	6	6	5	4
26	26	17	14	13	12	12	11	10	9	8	8	8	7	6	6	5
27	27	18	15	14	13	12	12	10	10	9	8	8	8	7	6	6
28	28	18	16	14	14	12	12	11	10	10	8	8	8	8	6	6
29	29	19	16	15	14	13	12	12	11	10	9	8	8	8	7	6
30	30	20	16	16	15	14	12	12	12	11	10	9	8	8	8	7
n/k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
31	31	20	17	16	16	15	13	12	12	12	11	10	9	8	8	8
32	32	21	18	16	16	16	14	13	12	12	12	10	10	8-9	8	8
33	33	22	18	16	16	16	14	14	12	12	12	11	10	9-10	8-9	8
34	34	22	19	17	16	16	15	14	13	12	12	12	10	10	9-10	8-9
35	35	23	20	18	16	16	16	15	14	12-13	12	12	11	10	10	9-10
36	36	24	20	18	17	16	16	16	14	13-14	12-13	12	12	11	10	10

II. Reed-Solomon Codes

MDS Codes

Proposition (Singleton Bound)

$$A_q(n,d) \leq q^{n-(d-1)}$$

Proof.

1. Let $C \subseteq \mathbb{F}_q^n$ have $|C| = A_q(n,d)$ and $d(C) \geq d$.
2. Write down all the $A_q(n,d)$ codewords.
3. Choose any $d-1$ coordinates and erase them.
4. Get $A_q(n,d)$ **distinct** elements of $\mathbb{F}_q^{n-(d-1)}$.

□

Definition

A code for which equality holds, $|C| = q^{n-(d-1)}$ is called Maximum Distance Separable or MDS.

Reed-Solomon Codes

Let $p_1, p_2, \ldots, p_q$ be an ordering of the elements of $\mathbb{F}_q$.
Let V_d be the vector space of polynomials in $\mathbb{F}_q[x]$ of degree at most d.

Definition

The evaluation map is defined by

$$\begin{aligned} \mathrm{ev}\colon \quad V_d &\mapsto \mathbb{F}_q^q \\ \mathrm{ev}(f) &= (f(p_1), \ldots, f(p_q)) \in \mathbb{F}_q^q. \end{aligned}$$

- $\mathrm{ev}(f+g) = \mathrm{ev}(f) + \mathrm{ev}(g)$ and $\mathrm{ev}(\alpha f) = \alpha\, \mathrm{ev}(f)$.
  The image $\mathrm{ev}(V_d) \subseteq \mathbb{F}_q^q$ is a linear code.

  It is the Reed-Solomon code of length q and order d, $\mathrm{RS}(q, d)$.
- As long as there is no nonzero polynomial vanishing at every element of $\mathbb{F}_q$, this map is injective, and $\dim(\mathrm{RS}(q,d)) = \dim(V_d) = d + 1$.

  $x^q - x$ vanishes at every element of $\mathbb{F}_q$, so suppose $q > d$.

Reed-Solomon Codes are MDS

Proposition

Let F be a field.
A nonzero $f \in F[x]$ with $\deg(f) = d$ has at most d distinct roots in F.

- Suppose $f, g \in \mathbb{F}_q[x]$ each have degree at most d.
  Then $f - g$ is either 0 or has at most d roots in $\mathbb{F}_q$.
- Conclude that $d(\mathrm{RS}(q,d)) = q - d$.
- $|\mathrm{RS}(q,d)| = q^{d+1} = q^{q-(d(\mathrm{RS}(q,d))-1)}$.
- Therefore, RS(q,d) is an MDS code.

Main Conjecture for MDS Codes

Definition

Let $M(k,q)$ be the maximum n such that a k-dimensional linear MDS code $C \subseteq \mathbb{F}_q^n$ exists.

Conjecture (Main Conjecture for MDS Codes)

1. *If $q \leq k$, $M(k,q) = k+1$. (Easy: Suppose now that $q > k$.)*
2. *If q is even and $k = 3$ or $k = q-1$, then $M(k,q) = q+2$.*
3. *Otherwise, $M(k,q) = q+1$.*

Reed-Solomon Example: For $q > d$, $M(d+1,q) \geq q$.

Reed-Solomon Code: Example

- Let $q = 5$, $d = 2$. Consider $\mathrm{RS}(5,2) \subseteq \mathbb{F}_5^5$.
- Choose a basis for polynomials in $\mathbb{F}_5[x]$ of degree at most 2: $1, x, x^2$.
- $\mathrm{RS}(5,2)$ is the row span of the generator matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 4 & 1 \end{pmatrix}$$

- No nonzero linear combination of rows has 0s in 3 or more coordinates.
- No 3×3 submatrix has determinant 0.

Question

- *Can we add another column to to get a 3×6 matrix over $\mathbb{F}_5$ such that no 3×3 submatrix has determinant 0?*
- *Is there a 3-dimensional MDS code $C \subseteq \mathbb{F}_5^6$ that gives $\mathrm{RS}(5,2)$ if you puncture in the last coordinate?*

Doubly Extended (Projective) Reed-Solomon Codes

Let

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$$

where each $a_i \in \mathbb{F}_q$.

Consider the map

$$\begin{aligned} \mathrm{ev}' : \quad V_d & \mapsto \mathbb{F}_q^{q+1} \\ \mathrm{ev}'(f) = & \ (f(p_1), \ldots, f(p_q), a_d) \,. \end{aligned}$$

- The image is a linear subspace of $\mathbb{F}_q^{q+1}$.
- If $q > d$ this map is injective and the dimension is $d+1$.
- The image is an MDS code.
  If f, g have the same x^d coefficient, then $\deg(f-g) \leq d-1$ and either $f - g = 0$ or $f - g$ has at most $d-1$ roots in $\mathbb{F}_q$.
- This is a Doubly Extended or Projective Reed-Solomon code.

Reed-Solomon Code: Example 2

- Let $q = 5, \; d = 2$. The doubly extended Reed-Solomon code is the row span of the generator matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 4 & 4 & 1 & 1 \end{pmatrix}$$

- No nonzero linear combination of rows has 0s in 3 or more coordinates.
- No 3×3 submatrix has determinant 0.
- This is a 3-dimensional MDS code $C \subset \mathbb{F}_5^6$.
- There is no 3-dimensional MDS code $C \subset \mathbb{F}_5^7$.
- $M(3, 5) = 6$.

1. A k-dimensional linear code $C \subseteq \mathbb{F}_q^n$ is the row span of a $k \times n$ generator matrix G.
2. C is an MDS code if and only if every nonzero linear combination of the rows of G has at most $k-1$ coordinates equal to 0.
3. Equivalently, no $k \times k$ submatrix of G has determinant 0.

- $M(k,q)$ is the maximum n such that a k-dimensional linear MDS code $C \subseteq \mathbb{F}_q^n$ exists.
- $M(k,q)$ is the maximum n for which there exists a $k \times n$ matrix with entries in $\mathbb{F}_q$ such that no $k \times k$ submatrix has determinant 0.

Main Conjecture for MDS Codes III

Definition

Let $M(k, q)$ be the maximum n such that a k-dimensional linear MDS code $C \subset \mathbb{F}_q^n$ exists.

Conjecture (Main Conjecture for MDS Codes)

1. *If $q \leq k$, $M(k, q) = k + 1$. (Easy: Suppose now that $q > k$.)*
2. *If q is even and $k = 3$ or $k = q - 1$, then $M(k, q) = q + 2$.*
3. *Otherwise, $M(k, q) = q + 1$.*

Doubly Extended Reed-Solomon codes give $M(d + 1, q) \geq q + 1$.
Ball: True for q prime.
Nathan's Favorite Matrix: 5-dimensional MDS code $C \subseteq \mathbb{F}_9^{10}$ that does not 'come from' a Reed-Solomon code [Glynn].

III. Projective Reed-Muller Codes

More Variables: Reed-Muller Codes

Definition

- *Choose an ordering of the points of* $\mathbb{F}_q^n : \; p_1, \ldots, p_{q^n}$.
- *Let* $V_{n,d}$ *be the* $\binom{n+d}{d}$*-dimensional vector space of polynomials in* $x_1, \ldots, x_n$ *of degree at most* d.
- *The evaluation map is defined by*

$$\begin{aligned} \mathrm{ev}: \quad V_{n,d} \quad &\mapsto \quad \mathbb{F}_q^{q^n} \\ \mathrm{ev}(f) = \quad &(f(p_1), \ldots, f(p_{q^n})) \in \mathbb{F}_q^{q^n} \end{aligned}$$

- *The image is a linear code.*
- *As long as there is no degree* d *polynomial vanishing at every element of* $\mathbb{F}_q^n$*, which is true for* $q > d$*, this map is injective and the image* $\mathrm{RM}_q(d, n)$ *had dimension* $\binom{n+d}{d}$.
- *Note that* $\mathrm{RM}_q(d, 1) = \mathrm{RS}(q, d)$.

Question

- *What is the minimum distance of* $\mathrm{RM}_q(d, n)$*?*
- *What is the maximum number of zeros of a polynomial of degree at most* d *in* $\mathbb{F}_q[x_1, \ldots, x_n]$*?*

- Let $\alpha_1, \ldots, \alpha_d$ be distinct elements of $\mathbb{F}_q$.

$$f(x_1, \ldots, x_n) = (x_1 - \alpha_1)(x_1 - \alpha_2) \cdots (x_1 - \alpha_d)$$

vanishes at $d \cdot q^{n-1}$ elements of $\mathbb{F}_q^n$.

- $d(\mathrm{RM}_q(d, n)) = q^n - dq^{n-1} = (q - d)q^{n-1}$.

For $n > 1$, these codes are very far from being MDS.

IV. Weight Enumerators of Reed-Muller Codes

The Hamming Weight Enumerator of a Code

Definition

The *Hamming weight enumerator* *of* $C \subseteq \mathbb{F}_q^n$ *is*

$$W_C(X, Y) = \sum_{c \in C} X^{n-\mathrm{wt}(c)} Y^{\mathrm{wt}(c)} = \sum_{i=0}^{n} A_i \cdot X^{n-i} Y^i,$$

where $A_i = \#\{c \in C \mid \mathrm{wt}(c) = i\}.$

Example

For $C = \{(0,0,0),(1,1,1)\} \subset \mathbb{F}_2^3$, $\quad W_C(X, Y) = X^3 + Y^3.$

Question

- *What is the weight enumerator of the Reed-Solomon code* $\mathrm{RS}(q, d)$*?*
- *How many* $f \in \mathbb{F}_q[x]$ *of degree at most* d *have exactly* m *distinct roots in* $\mathbb{F}_q$*?*

◦ **Fact**: The weight enumerator of a k-dimensional MDS code $C \subseteq \mathbb{F}_q^n$ is determined by its parameters.

- Computing the weight enumerator of $\mathrm{RM}_q(1, n)$ is easy.
- Computing the weight enumerator of $\mathrm{RM}_q(2, n)$ is a counting problem about quadratic forms over finite fields.

Proposition

We have that $W_{\mathrm{RM}_q(2,2)}(X, Y)$ *is equal to*

$$X^{q^2} + \frac{(q-1)(q^3-q+2)}{2} Y^{q^2} + \frac{(q-1)^2 q^3}{2} X Y^{q^2-1}$$
$$+ \frac{(q-1)^2 q^3 (q+1)}{2} X^{q-1} Y^{q^2-q+1} + (q^3-q)(q^2-q+2) X^q Y^{q^2-q}$$
$$+ \frac{(q-1)^3 q^3}{2} X^{q+1} Y^{q^2-q-1} + \frac{(q-1)(q+1)q^3}{2} X^{2q-1} Y^{q^2-2q+1}$$
$$+ \frac{q(q+1)(q-1)^2}{2} X^{2q} Y^{q^2-2q}.$$

Reed-Muller Codes from Cubic Curves

Question

1. *How many $f_3 \in \mathbb{F}_q[x, y]$ of degree at most 3 have exactly m zeros?*
2. *How many smooth cubic curves $\{f_3(x, y) = 0\}$ have exactly m $\mathbb{F}_q$-rational points?*

A smooth cubic curve with an $\mathbb{F}_q$-rational point defines an elliptic curve.

Question

1. *How many isomorphism classes of elliptic curves $E/\mathbb{F}_q$ have a given number of $\mathbb{F}_q$-points?*
2. *For how many $a, b \in \mathbb{F}_q$ does the equation $y^2 = x^3 + ax + b$ have exactly m solutions $(x, y) \in \mathbb{F}_q^2$?*

Deuring, Waterhouse: Answer involves class numbers of orders in imaginary quadratic fields.

Put this together and get $W_{\mathrm{RM}_q(3,2)}(X, Y)$.

Question

1. *How many* $f_4 \in \mathbb{F}_q[x, y]$ *of degree at most 4 have exactly m zeros?*
2. *How many smooth quartic curves* $\{f_4(x, y) = 0\}$ *have exactly m* $\mathbb{F}_q$*-rational points?*
3. *What is the maximum number of* $\mathbb{F}_q$*-points of a smooth quartic curve* $\{f_4(x, y) = 0\}$*?*

Question

Can we say statistical things about the coefficients of $W_{\mathrm{RM}_q(2,4)}(X, Y)$*?*

The coefficients of $W_{\mathrm{RM}_q(2,3)}(X, Y)$ have a symmetry that the coefficients of $W_{\mathrm{RM}_q(2,4)}(X, Y)$ no longer have...

Definition

Let $N_q(t)$ be the number of $\mathbb{F}_q$-isomorphism classes of smooth (projective) plane quartics with $\#C(\mathbb{F}_q) = q + 1 - t$, each class weighted by $\frac{1}{\#\mathrm{Aut}_{\mathbb{F}_q}(C)}$.

For $0 \leq t \leq 6\sqrt{q}$, let

$$\mathcal{V}_q(t) := N_q(t) - N_q(-t).$$

Not true that $N_q(t)$ must equal $N_q(-t)$.

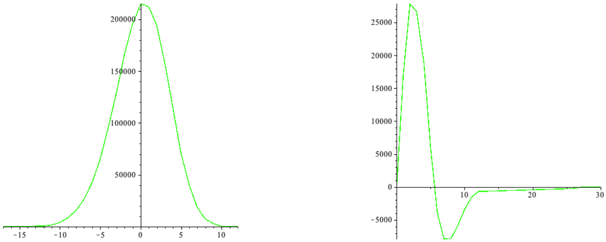


Figure: Graphs of $N_{11}(t)$ and $\mathcal{V}_{11}(t)$

See work of Lercier, Ritzenthaler, Rovetta, Sijsling, and Smith.

The Dual Code of a Linear Code

Definition

1. *For* $\substack{x=(x_1,\ldots,x_n)\\ y=(y_1,\ldots,y_n)} \in \mathbb{F}_q^n$ *let* $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$.
2. *For a linear code* $C \subseteq \mathbb{F}_q^n$*, the* *dual code* *is defined by*

$$C^{\perp} = \left\{ y \in \mathbb{F}_q^n \mid \langle x, y\rangle = 0 \ \forall x \in C \right\}.$$

Example

Let $C = \{(0,\ldots,0),(1,\ldots,1)\} \subset \mathbb{F}_2^n$.
Then $C^{\perp} = \{y \in \mathbb{F}_2^n \mid \text{wt}(y) \text{ is even}\}$.
We see that

$$W_C(X,Y) = X^n + Y^n,$$

and

$$W_{C^{\perp}}(X,Y) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} X^{n-2i} Y^{2i} = \frac{(X+Y)^n + (X-Y)^n}{2}.$$

Theorem (MacWilliams)

For a linear code $C \subseteq \mathbb{F}_q^n$

$$W_{C^\perp}(X, Y) = \frac{1}{|C|} W_C\left(X + (q-1)Y, X - Y\right).$$

◦ One way to prove this involves discrete Poisson summation.

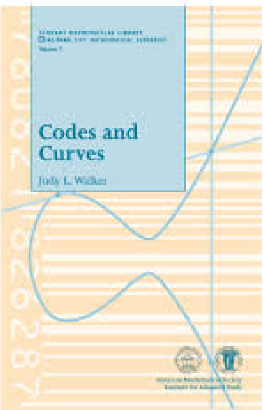
Idea: Study the weight enumerator of a code C by studying the weight enumerator of its dual code $C^\perp$.

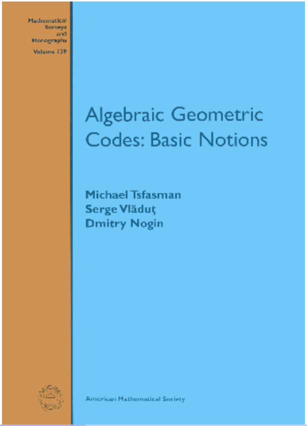
V. What else is there?

Algebraic Geometry Codes

Idea: Take a vector space of polynomials V. Get a code by evaluating elements of V at some subset of points of $\mathbb{F}_q^n$.

$\circ$ Number Theory $\rightarrow$ Coding Theory. Construct 'good codes' from Riemann-Roch spaces of divisors of algebraic curves with many $\mathbb{F}_q$-points.





Suppose you have a good code $C \subseteq \mathbb{F}_q^n$.

Question

How do you construct an ***efficient*** *encoding/decoding scheme?*

Question

I send you a message. You receive something that is not in the code. How do you find the codeword closest to it?