Math 206A: Algebra

Midterm 2

Monday, November 23, 2020.

- You have **90 minutes** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

Problems	
1 (5 Points)	
2 (5 Points)	
3 (10 Points)	
4 (10 Points)	
5 (10 Points)	
Total	

Problems	
6 (10 Points)	
7 (10 Points)	
8 (10 Points)	
9 (5 Points)	
10 (10 Points)	
Total	

Problems

1. Suppose G is an abelian group, and H_1, H_2 are subgroups.
Either prove the following statement or find a counterexample.

$$\text{If } G/H_1 \cong G/H_2, \quad \text{then } H_1 \cong H_2.$$

2. State whether the following statement is true or false.
Give a 1-2 sentence explanation for your answer.

Every finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$ is abelian.

3. (a) Describe all abelian groups of order 64 up to isomorphism.
(That is, give a list of abelian groups of order 64 such that every abelian group of order 64 is isomorphic to exactly one in your list.)
   (b) Is $\mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/48\mathbb{Z}$ isomorphic to $\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/60\mathbb{Z}$?
Prove your answer is correct.

4. Let H, K be two subgroups of a finite group G such that $H \cap K = \{1\}$ and $|H| \cdot |K| = |G|$. Does it follow that $G \cong H \times K$?
Either prove this statement or give a counterexample.

5. Let $\varphi\colon G \to H$ be a surjective homomorphism between finite groups.
Prove that the image of a Sylow p-subgroup in G will be a Sylow p-subgroup in H.

6. A subgroup H of a group G is characteristic if for every $\sigma \in \mathrm{Aut}(G)$, $\sigma(H) = H$.
Prove that every subgroup of a cyclic group is characteristic.

7. Let G be a group and let $[G, G]$ denote its commutator subgroup. Suppose that $H \leq G$ satisfies $[G, G] \leq H$. Prove that H is a normal subgroup of G.

8. Suppose that G is a group of order $351 = 3^3 \cdot 13$. Prove that G is not simple.

9. Give an example of an infinite group G in which every element of G has finite order.
No explanation is needed– you just need to give the example.

10. (a) Prove that Q_8 is not isomorphic to a semidirect product of two groups of order smaller than 8.
    (b) Is D_8 isomorphic to a semidirect product of two groups of order smaller than 8?
If so, then give groups H, K and a homomorphism φ such that $D_8 \cong H \rtimes_\varphi K$.
If not, prove the D_8 is not isomorphic to a semidirect product of two groups of order smaller than 8.