Math 206A: Algebra
Homework 3

Due Friday, October 30th at 12PM.
Please email `nckaplan@math.uci.edu` with questions.

One goal of this assignment is to help you prepare for Midterm 1. (For more information on Midterm 1, see the ‘Midterm 1’ page under the ‘Pages’ tab on the course website.) We have not yet had any HW problems on Group Actions, and you should expect this topic to be a significant part of the exam. Because of this, we have included some exercises from Chapter 4 of Dummit and Foote. The rest the of assignment consists of some UCI Algebra Comprehensive Exam problems.

1. Exercise 7 of Section 4.2. Given a finite group G of order n Cayley's theorem tells us that it is embedded as a subgroup of S_n. This exercise gives one example where you determine whether we can actually find it as a subgroup of a smaller symmetric group.
2. Exercise 11 of Section 4.2. The left regular representation gives a homomorphism from G to S_G. This problem says something about the permutations we get in the image.
3. Exercise 5 of Section 4.3. This exercise gives a nice relation between the center of a group G and the sizes of the conjugacy classes in G.
4. Spring 2019 Comprehensive Exam #2: Let G be a finite simple groups having a subgroup H of prime index p. Prove that p is the largest prime divisor of $|G|$.
5. Spring 2017 Comprehensive Exam #1: Does there exist an integer $n \geq 4$ such that the symmetric group S_n is isomorphic to a dihedral group? Prove your answer.
6. Spring 2015 Comprehensive Exam #2: Show that is G is a finite group of odd order and $N \trianglelefteq G$ is a normal subgroup of order 5, then N is in the center of G.
7. Spring 2014 Comprehensive Exam #1: Suppose G is a finite group, H is a subgroup of G, and $a \in G$ is an element such that $a^k \in H$ for some integer k with $\gcd(k, |G|) = 1$. Prove that $a \in H$.
8. Spring 2010 Comprehensive Exam #3: Let A be the group of rationals under addition and M the group of non-zero rationals under multiplication. Determine all homomorphisms $\phi \colon A \to M$.
9. Spring 2007 Comprehensive Exam #3 (small modification): Prove that a group G is infinite if and only if it has infinitely many subgroups.
10. Spring 2006 Comprehensive Exam #3: Let $\pi = \{p_1, \dots, p_m\}$ be a nonempty set of primes. A π**-group** is a group whose order n has the property that all primes dividing n lie in the set π. For example, a group of order $2^3 \cdot 5 \cdot 7^2$ is a $\{2, 5, 7\}$-group.

    Let G be a finite group and let H, K be normal subgroups of G such that G/H and G/K are π-groups. Prove that $G/(H \cap K)$ is also a π-group.
    Hint: Use the isomorphism theorems.