Math 206A: Algebra
Homework 6

Due Friday, November 20th at 12PM.
Please email `nckaplan@math.uci.edu` with questions.

1. Exercise 5 of Section 5.4.
   We have encountered the commutator subgroup of a group in some earlier homework exercises. This is discussed in more detail at the start of Section 5.4. I did not cover this material in lecture. In this exercise you compute the commutator subgroup of S_n.

2. Exercise 14 of Section 5.4.
   In this exercise you apply the recognition theorem for direct products to a group of matrices.

3. Exercise 6 of Section 5.5.
   This exercise plays an important role in showing that certain semidirect products are isomorphic to each other. This exercise gets used frequently at the end of Section 5.5 where semidirect products are used to classify groups of given order.
   This statement is also given in a slightly different form in Lemma 6.2 of Conrad's 'Semidirect Product' notes. (It's an exercise there too.)

4. Exercise 8 of Section 5.5.
   In this exercise you classify non-abelian groups of order 75 using Exercise 6.

5. In Lecture 22 we discussed Example (7) from page 179 of Dummit and Foote, where we saw how to construct two non-isomorphic non-abelian groups of order p^3, where p is an odd prime. The first of these two constructions gives a group isomorphic to the Heisenberg group over $\mathbb{Z}/p\mathbb{Z}$. This can be deduced from the classification of groups of order p^3 on page 183-184, but this argument is a little unsatisfying because it's not so clear how the semidirect product gives this group of matrices. In this exercise, you will work out this isomorphism explicitly.

   (a) Throughout this problem let p be an odd prime.
   Recall that the Heisenberg group over $\mathbb{Z}/p\mathbb{Z}$ is
   $$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}/p\mathbb{Z} \right\}.$$
   We have seen earlier that this is a non-abelian group of order p^3.
   Prove that the subset of elements of the form
   $$\left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : b, c \in \mathbb{Z}/p\mathbb{Z} \right\}$$
   is a subgroup of G isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Is this a normal subgroup?

   (b) Prove that the subset of elements of the form
   $$\left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a \in \mathbb{Z}/p\mathbb{Z} \right\}$$
   is a subgroup of G isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Is this a normal subgroup?

(c) Show that G is isomorphic to the semidirect product $(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \rtimes_\varphi \mathbb{Z}/p\mathbb{Z}$.
Explicitly describe the homomorphism $\varphi\colon \mathbb{Z}/p\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$.
Explicitly describe an isomorphism between these groups.
(Let $(h, k) \in (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \rtimes_\varphi \mathbb{Z}/p\mathbb{Z}$. Which matrix in G are you sending it to?)

(d) Prove that the Heisenberg group over $\mathbb{Z}/2\mathbb{Z}$ is isomorphic to D_8.

6. In Lecture 22 we stated that every non-abelian group of order 12 is isomorphic to either D_{12}, A_4, or the semidirect product $\mathbb{Z}/3\mathbb{Z} \rtimes_\varphi \mathbb{Z}/4\mathbb{Z}$ where $\varphi\colon \mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/3\mathbb{Z})$ is a non-trivial homomorphism. (See Examples (1) and (2) on page 178.) This is proven on pages 182-183, and also (in a different way) in these notes of Conrad on 'Groups of Order 12': `https://kconrad.math.uconn.edu/blurbs/grouptheory/group12.pdf`.

   Consider the group

$$\mathrm{Aff}(\mathbb{Z}/6\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in (\mathbb{Z}/6\mathbb{Z})^*,\ b \in \mathbb{Z}/6\mathbb{Z} \right\}.$$

   This is a non-abelian group of order 12.
   Which of the groups given above is it isomorphic to?

7. 2008 Algebra Advisory Exam #3: Let G be a group of order $p^a m$ where p is a prime with $p \nmid m$. Let P be a Sylow p-subgroup of G. Prove that if $P = N_G(P)$, then $n_p = m$.

8. 2010 Algebra Advisory Exam #1: Show that any group of order 185 is abelian.

9. Prove that if G is a group in which every non-identity element has order 2, then G is abelian.