Math 206A: Algebra Summary: Rings

In this document, we'll give a list of some of the highlights that we covered in the last part of the course. This is meant to be a guide to help you study for the Final Exam. We will include a separate document focusing on groups.

1 Ring Theory Basics

1.1 Definitions

1. Ring;
2. Field;
3. Division Ring/Skew Field;
4. Zero Divisor;
5. Unit;
6. Integral Domain;
7. Commutative Ring;
8. Unital Ring (Ring with Identity);
9. Subring;
10. Quotient Ring;
11. Left Ideal, Right Ideal, (Two-Sided) Ideal;
12. Ring Homomorphism;
13. Kernel and Image of a Ring Homomorphism;
14. The sum and product of ideals;
15. Ideal Generated by a Set;
16. Principal Ideal;
17. Maximal Ideal;
18. Prime Ideal.

1.2 Examples

1. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$;
2. $\mathbb{Z}/n\mathbb{Z}$;
3. Rings of Functions;
4. Real (Hamilton) Quaternions;
5. Quadratic Fields $\mathbb{Q}(\sqrt{D})$ and Quadratic Integer Rings;
6. Polynomial Rings $R[x], R[x_1, \ldots, x_n]$;
7. Matrix Rings $M_n(R)$;
8. Example of an ideal that is not principal;
9. Example of an ideal that is not finitely generated;
10. Example of a prime ideal that is not maximal.

1.3 Theorems

1. Let $a, b, c \in R$ with a not a zero divisor. If $ab = ac$ either $a = 0$ or $b = c$. If R is an integral domain, then $ab = ac$ implies $a = 0$ or $b = c$.
2. A finite integral domain is a field.
3. The units in $R[x]$ are the units of R. R is an integral domain if and only if $R[x]$ is an integral domain.
4. If R is a ring and I is an ideal of R, then the quotient group R/I is a ring with multiplication operation $(r + I) \cdot (s + I) = rs + I$. Conversely, if I is an additive subgroup of R such that the operation above is well-defined, then I is an ideal of R.
5. The kernel of a ring homomorphism is an ideal. Every ideal I is the kernel of the natural projection homomorphism from R to R/I.
6. Isomorphism Theorems for Rings (1st, 2nd, 3rd, 4th).
7. A commutative ring R with identity $1 \neq 0$ if a field if and only if R has no nonzero proper ideals.
8. In a commutative ring R with identity $1 \neq 0$, every proper ideal is contained in a maximal ideal. (Proof used Zorn's lemma.)
9. In a commutative ring R with identity $1 \neq 0$, if M is an ideal of R then M is maximal if and only if R/M is a field.

10. In a commutative ring R with identity $1 \neq 0$, if P is an ideal of R then P is prime if and only if R/P is an integral domain.

11. In a commutative ring R with identity $1 \neq 0$, every maximal ideal is prime.